# *Guide to* 
# Mechanical Finite Element Modelling using FEBio

Jef Aernouts – Laboratory of Biomedical Physics – Last updated: 25/10/2011

## *1. Introduction*

In the Laboratory of Biomedical Physics (BIMEF), we are –among other things– interested in the mechanical behaviour of biological structures. Examples are the motion and deformation of the mammalian tympanic membrane, stress analysis in various evolutionary adapted bird beaks and the mechanical behaviour of blood vessels during blood transport.

The physical theory that describes the phenomena of internal forces (stress) and deformations (strain) is called elasticity theory. It establishes a mathematical model that allows solving various mechanical problems.

The equations describing elastic behaviour can be solved using finite element modelling. This is a numerical technique that gives approximate solutions. The finite element method becomes very useful when exact analytical solutions of a problem cannot be found.

In this short guide, a brief introduction with useful references to the theory of elasticity and the theory of finite element modelling is given. Furthermore the finite element software that we use in our laboratory, called FEBio (Finite Elements for Biomechanics), is introduced. The references are attached in order of appearance at the end of this document.

## *2. Theory of Elasticity*

The concept of the elastic force-deformation relation was first proposed by Robert Hooke in 1678. However, the major formulation of the mathematical theory of elasticity was not developed until the 19$^{th}$ century.

As a result of applied loadings, elastic solids will change shape or deform. An elastic solid is said to be deformed or strained when the *relative* displacements between points in the body are changed. This is in contrast to rigid body motion, where the distance between points remains the same. The work done by the applied loadings is stored inside the body in the form of strain energy. For an idealized body, this stored energy is completely recoverable when the solid is returned to its original configuration.

This section is divided in two parts. Section 2.1 introduces linear elasticity theory, which is valid as long as the deformations are small. In many problems, however, deformations become large and non-linear models should be considered. The theory of non-linear solid mechanics is introduced in section 2.2.

### 2.1 Linear elastic materials

The development of the basic field equations of elasticity theory begins with a description of the kinematics of material deformation. When the shape of an elastic body changes, this deformation can be quantified by knowing the displacements of all material points in the body. The description of the deformation of an infinitesimal part in the continuum leads to the development of the strain tensor. When the deformations are small, i.e. in the case that linear elasticity can be used, the *small strain* tensor is a good descriptor. The small strain tensor is introduced in Appendix A (2.1-2.2).

When a structure is subjected to applied external loadings, internal forces are introduced in the body. These internal forces are distributed continuously within the continuum. The *Cauchy stress* tensor describes the internal forces per unit area and is introduced in Appendix B (3.1-3.2).
Up till now, we have not considered specific material response. We only introduced descriptions for deformations and internal forces. However, it is to be expected that steel will for example behave stiffer than aluminum. To conclude this section, a particular material model that provides reasonable characterization of materials under small deformations is specialized: the *linear elastic material*, see Appendix C (4.1-4.3). In linear elasticity theory the fourth order elasticity tensor $\mathbf{C}$ linearly relates the small strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ as follows: $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$ (index notation: $\sigma_{ij} = \mathsf{C}_{ijkl} \cdot \varepsilon_{kl}$). When the material is isotropic, only two independent elastic constants are needed to describe the behaviour: *Young's modulus E* and *Poisson's ratio* $\upsilon$.
The easiest way to experimentally characterize materials is performing a uniaxial tension test in which a cylindrical or flat sample is loaded axially. The axial strain in this case is the relative change in length, the stress is the measured force divided by the cross-sectional area. For a linear material, the relationship between uniaxial strain and stress is linear. This is for example true for steel under small deformations, typical up till 1%.

## 2.2 Non-linear solid mechanics

Performing a uniaxial tension test on a rubber sample, let's say up till 100%, one will see that the stress-strain curve is highly non-linear. This is an example of a situation where non-linear continuum mechanics must be used.
Two sources of non-linearity exist in the analysis of solid continua, namely material and geometric non-linearity. The former occurs when, for whatever reason, the stress-strain behaviour is non-linear. The latter is important when changes in geometry have an effect on the load deformation behaviour.
For *large* –also denoted with *finite*– deformations (in contrast to small deformations), the undeformed and deformed configurations can be significantly different and a distinction between these two configurations must be maintained. This gives rise to another description of strain as compared to linear elasticity.
A key quantity in finite deformation analysis is the *deformation gradient* **F**, which is central to the description of (large deformation) strain. Consider the deformation of an object when it moves from the initial or *reference* configuration (denoted with material coordinates **X**) to the *current* configuration (denoted with spatial coordinates **x**). The *deformation map* $\phi$ maps the coordinates of a material point to the spatial configuration: $\mathbf{x}=\phi(\mathbf{X})$. The deformation gradient tensor is now defined as $\mathbf{F} = \frac{\partial \phi}{\partial \mathbf{X}}$, and relates an infinitesimal vector in the reference configuration $d\mathbf{X}$ to the corresponding vector in the current configuration $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$. In order to define a strain measure, we first introduce the *right Cauchy-Green deformation tensor* as follows: $\mathbf{C}=\mathbf{F}^T\mathbf{F}$. This tensor can now be used to define an appropriate strain measure in the material configuration, the *Green-Lagrange strain tensor*: $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, with $\mathbf{I}$ the identity tensor. A thorough and mathematical description of the previous matter is given in Appendix D (3.1-3.5). It is however noted that this may be –in the scope of the student's work– too exhaustive.

In section 2.1 about linear elasticity we defined the Cauchy stress tensor ($\boldsymbol{\sigma}$), which is a spatial tensor since it is described in the current configuration. Since the Green-Lagrange strain tensor is defined in material coordinates, we need a material stress measure associated with the initial configuration of the body. This leads to the *second Piola-Kirchhoff stress tensor*, given as $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$ ($J$ = determinant of deformation gradient tensor). In Appendix E (4.1-4.5), more info can be found. Again, this may be too exhaustive.
We now arrive at the point in which we want to find expressions that correlate stress and strain. These expressions, known as constitutive equations, obviously depend on the type of material. For example classical small strain elasticity involved a Young's modulus and a Poisson's ratio and related stress and strain linearly. To allow a more general behaviour, we now show the concept of a hyperelastic material whereby stresses are derived from a stored elastic energy function.
When the work done by the stresses during a deformation process is dependent only on the initial state and the final configuration, the behaviour of the material is termed *hyperelastic*. As a consequence a *stored energy function* or *elastic potential* per unit undeformed volume can be established which is only dependent of the current deformation gradient, so that we have for the energy density function: $\Psi(\mathbf{F}(\mathbf{X}), \mathbf{X})$. The dependency upon $\mathbf{X}$ allows for possible inhomogeneity. For convenience, $\Psi$ is often expressed as a function of $\mathbf{C}=\mathbf{F}^T\mathbf{F}$, thus $\Psi(\mathbf{C}(\mathbf{X}), \mathbf{X})$.
For the very important case of *isotropic* materials, it is required that the constitutive behaviour is independent of the material axes chosen. In general, the components of a second-order tensor will change when the axes are rotated. The *invariants* of a second-order tensor however remain unaltered under such transformations. Consequently, for isotropic materials $\Psi$ must only be a function of the invariants of $\mathbf{C}$. The invariants, denoted $I_1, I_2, I_3$, are given as $I_1$ = tr $\mathbf{C}$, $I_2 = ½[(\text{tr } \mathbf{C})^2 - \text{tr } \mathbf{C}^2]$, $I_3$ = det $\mathbf{C}$. For isotropic materials we hence write $\Psi(I_1, I_2, I_3, \mathbf{X})$.
A material is considered incompressible if it shows no change in volume during deformation. Rubber materials show for example nearly-incompressible behaviour under typical, non-hydrostatic loadings. Incompressibility means that $J = 1$ (no volume change) holds throughout the entire body. When one wants to model incompressible behaviour, it is useful to separate the volumetric and the deviatoric (distortional) components of the deformation gradient. Using this separation, the energy density function $\Psi$ can be given as $\Psi(\mathbf{C}) = \Psi(\hat{\mathbf{C}}) + U(J)$. In this equation, $\Psi(\hat{\mathbf{C}})$ is the distortional component of the energy function in which no volumetric effects are included. $U(J)$ represents the volumetric energy component. An example for $U$ that is used in the following definitions of constitutive models is $U(J) = ½\,\kappa\,(\ln J)^2$. $\kappa$ represents the bulk modulus, a high value of it will enforce incompressibility since it will enforce $J = 1$. A detailed description about hyperelasticity can be found in Appendix F.
We end this section by giving two frequently used hyperelastic materials: the Mooney-Rivlin and the Ogden material. The *Mooney-Rivlin material* is given as:

$$\Psi = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + \frac{1}{2}\kappa(\ln J)^2,$$

with $C_{10}$ and $C_{01}$ the Mooney-Rivlin material constants that specify material behaviour.

Another commonly used hyperelastic material was proposed by Ogden. His energy density function is given in terms of the eigenvalues of the deviatoric part of the right Cauchy-Green deformation tensor ($\lambda_{1,}\ \lambda_{2,}\ \lambda_3$) which are also independent of the chosen material axes:

$$\Psi = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3) + \frac{1}{2}\kappa(\ln J)^2.$$

$\mu_i$ and $\alpha_i$ are the material constants, *N* is the order of the model.

## 3. Finite element modelling

The finite element method has been developed over the last 40 years into a popular technique for solving a number of significant problems in engineering and physics. For mechanical problems, it will predict displacements and stresses.

The method dicretizes the domain under study, i.e. the material geometry, by dividing the region into subdomains called *elements*. This process is called *meshing*. Within each element, an approximate solution is developed, and this is quantified at particular locations called *nodes*. Then, based on system connectivity the elements are assembled and boundary conditions are applied. This results in an algebraic set of –in general non-linear– equations which can be solved, usually with the Newton-Raphson iterative technique. Because the element size can be varied, the method can accurately simulate problems of complex geometry and loadings.

Generally, the finite element formulation is established in terms of a weak (variational) form of the differential equations under consideration. In the context of solid mechanics this implies the use of the *virtual work* equation. This equation states that the sum of the work done by applied forces and the internal energy for virtual displacements is zero. This is a variational approach to find the minimum of the total potential energy. In a dynamic (time-dependent) analysis, the inertial effects are also included in the equations. This is important when modelling dynamic loadings, e.g. deformation of the tympanic membrane under acoustic frequencies.

In Appendix G the finite element formulas derivations are given for two-dimensional, linear isotropic elastostatic problems. For general, three-dimensional non-linear time-dependent problems, the theory is more exhaustive and not given here. However, the limited case in Appendix G will give useful insight in the procedure.

## 4. FEBio

*FEBio* (Finite Elements for Biomechanics) is a non-linear finite element solver that is specifically designed for biomechanical applications. It offers modelling scenarios, constitutive models and boundary conditions that are relevant to many research areas in biomechanics.

FEBio supports two analysis types, namely quasi-static and dynamic. In a quasi-static analysis, the (quasi-) static response of the system is sought and the effects of inertia are ignored. In a dynamic analysis, the inertial effects are included in the governing equations to calculate the time dependent response of the system.

FEBio runs on several different computing platforms including Windows XP, Mac OSX and most versions of Linux. FEBio is started from a shell window (also known as the command prompt in Windows). The name of an input file (*.feb*) has to be specified in the command line.

FEBio does not have mesh generation capabilities. Therefore the input files need to be generated by pre-processing software. The pre-processor associated with FEBio is called *PreView*.
After running FEBio, two files are created: the log file (*.log*) and the plot file (*.plt*). The log file is a text file that contains the screen output that was generated during a run.The user can request FEBio to output additional data in the log file. This is very useful to extract nodal positions, forces … The plot file contains the results of the analysis. Since this is a binary file, the results must be analyzed using the post processing software *PostView*.
FEBio and the related PreView and PostView software packages can be downloaded at the following link: http://mrl.sci.utah.edu/software. First create an account, next you can start downloading. The user's manuals of FEBio, PreView and PostView provide a good basis to become familiar with the software package. The PreView user's manual shows some easy problems to start with. They are also available for download at the website.

## *5. Appendices*

- Appendix A – Deformation: Displacements and Strains
  *From: Elasticity. Theory, Applications and Numerics – Martin H. Sadd*

- Appendix B – Stress and Equilibrium
  *From: Elasticity. Theory, Applications and Numerics – Martin H. Sadd*

- Appendix C – Material Behaviour – Linear Elastic Solids
  *From: Elasticity. Theory, Applications and Numerics – Martin H. Sadd*

- Appendix D – Kinematics
  *From: Nonlinear continuum mechanics for finite element analysis - Bonet and Wood*

- Appendix E – Stress and Equilibrium
  *From: Nonlinear continuum mechanics for finite element analysis - Bonet and Wood*

- Appendix F – Hyperelasticity
  *From: Nonlinear continuum mechanics for finite element analysis - Bonet and Wood*

- Appendix G – Stress and Equilibrium
  *From: Elasticity. Theory, Applications and Numerics – Martin H. Sadd*

# Appendix

# A

# 2 Deformation: Displacements and Strains

We begin development of the basic field equations of elasticity theory by first investigating the kinematics of material deformation. As a result of applied loadings, elastic solids will change shape or deform, and these deformations can be quantified by knowing the displacements of material points in the body. The continuum hypothesis establishes a displacement field at all points within the elastic solid. Using appropriate geometry, particular measures of deformation can be constructed leading to the development of the strain tensor. As expected, the strain components are related to the displacement field. The purpose of this chapter is to introduce the basic definitions of displacement and strain, establish relations between these two field quantities, and finally investigate requirements to ensure single-valued, continuous displacement fields. As appropriate for linear elasticity, these kinematical results are developed under the conditions of small deformation theory. Developments in this chapter lead to two fundamental sets of field equations: the strain-displacement relations and the compatibility equations. Further field equation development, including internal force and stress distribution, equilibrium and elastic constitutive behavior, occurs in subsequent chapters.

## 2.1 General Deformations

Under the application of external loading, elastic solids deform. A simple two-dimensional cantilever beam example is shown in Figure 2-1. The undeformed configuration is taken with the rectangular beam in the vertical position, and the end loading displaces material points to the deformed shape as shown. As is typical in most problems, the deformation varies from point to point and is thus said to be *nonhomogenous*. A superimposed square mesh is shown in the two configurations, and this indicates how elements within the material deform locally. It is apparent that elements within the mesh undergo extensional and shearing deformation. An elastic solid is said to be deformed or strained when the *relative displacements* between points in the body are changed. This is in contrast to *rigid-body motion* where the distance between points remains the same.

In order to quantify deformation, consider the general example shown in Figure 2-2. In the undeformed configuration, we identify two neighboring material points $P_o$ and P connected with the *relative position vector* $\boldsymbol{r}$ as shown. Through a general deformation, these points are mapped to locations $P'_o$ and $P'$ in the deformed configuration. For *finite* or *large deformation theory,* the

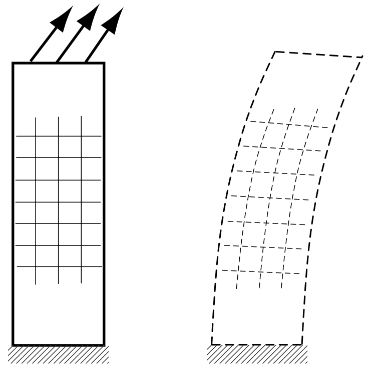



**FIGURE 2-1** *Two-dimensional deformation example.*

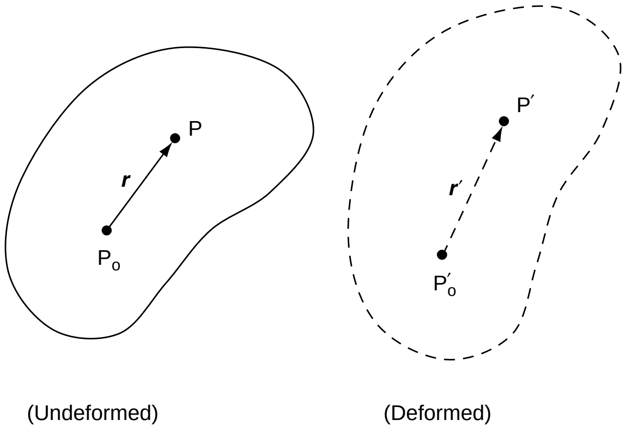



**FIGURE 2-2** *General deformation between two neighboring points.*

undeformed and deformed configurations can be significantly different, and a distinction between these two configurations must be maintained leading to Lagrangian and Eulerian descriptions; see, for example, Malvern (1969) or Chandrasekharaiah and Debnath (1994). However, since we are developing linear elasticity, which uses only small deformation theory, the distinction between undeformed and deformed configurations can be dropped.

Using Cartesian coordinates, define the displacement vectors of points $P_o$ and P to be $\boldsymbol{u^o}$ and $\boldsymbol{u}$, respectively. Since P and $P_o$ are neighboring points, we can use a Taylor series expansion around point $P_o$ to express the components of $\boldsymbol{u}$ as

$$
\begin{aligned}
u &= u^o + \frac{\partial u}{\partial x} r_x + \frac{\partial u}{\partial y} r_y + \frac{\partial u}{\partial z} r_z \\
v &= v^o + \frac{\partial v}{\partial x} r_x + \frac{\partial v}{\partial y} r_y + \frac{\partial v}{\partial z} r_z \\
w &= w^o + \frac{\partial w}{\partial x} r_x + \frac{\partial w}{\partial y} r_y + \frac{\partial w}{\partial z} r_z
\end{aligned}
\tag{2.1.1}
$$

Note that the higher-order terms of the expansion have been dropped since the components of $\boldsymbol{r}$ are small. The change in the relative position vector $\boldsymbol{r}$ can be written as

$$\Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r} = \boldsymbol{u} - \boldsymbol{u}^o \tag{2.1.2}$$

and using (2.1.1) gives

$$\begin{aligned}
\Delta r_x &= \frac{\partial u}{\partial x} r_x + \frac{\partial u}{\partial y} r_y + \frac{\partial u}{\partial z} r_z \\
\Delta r_y &= \frac{\partial v}{\partial x} r_x + \frac{\partial v}{\partial y} r_y + \frac{\partial v}{\partial z} r_z \\
\Delta r_z &= \frac{\partial w}{\partial x} r_x + \frac{\partial w}{\partial y} r_y + \frac{\partial w}{\partial z} r_z
\end{aligned} \tag{2.1.3}$$

or in index notation

$$\Delta r_i = u_{i,j} r_j \tag{2.1.4}$$

The tensor $u_{i,j}$ is called the *displacement gradient tensor,* and may be written out as

$$u_{i,j} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix} \tag{2.1.5}$$

From relation (1.2.10), this tensor can be decomposed into symmetric and antisymmetric parts as

$$u_{i,j} = e_{ij} + \omega_{ij} \tag{2.1.6}$$

where

$$\begin{aligned}
e_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\
\omega_{ij} &= \frac{1}{2}(u_{i,j} - u_{j,i})
\end{aligned} \tag{2.1.7}$$

The tensor $e_{ij}$ is called the *strain tensor,* while $\omega_{ij}$ is referred to as the *rotation tensor*. Relations (2.1.4) and (2.1.6) thus imply that for small deformation theory, the change in the relative position vector between neighboring points can be expressed in terms of a *sum* of strain and rotation components. Combining relations (2.1.2), (2.1.4), and (2.1.6), and choosing $r_i = dx_i$, we can also write the general result in the form

$$u_i = u_i^o + e_{ij} dx_j + \omega_{ij} dx_j \tag{2.1.8}$$

Because we are considering a general displacement field, these results include both strain deformation and rigid-body motion. Recall from Exercise 1-14 that a dual vector $\omega_i$ can be associated with the rotation tensor such that $\omega_i = -1/2\varepsilon_{ijk}\omega_{jk}$. Using this definition, it is found that

$$\begin{aligned}
\omega_1 = \omega_{32} &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\
\omega_2 = \omega_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\
\omega_3 = \omega_{21} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)
\end{aligned} \tag{2.1.9}$$

which can be expressed collectively in vector format as $\boldsymbol{\omega} = (1/2)(\nabla \times \boldsymbol{u})$. As is shown in the next section, these components represent rigid-body rotation of material elements about the coordinate axes. These general results indicate that the strain deformation is related to the strain tensor $e_{ij}$, which in turn is a related to the displacement gradients. We next pursue a more geometric approach and determine specific connections between the strain tensor components and geometric deformation of material elements.

## 2.2 Geometric Construction of Small Deformation Theory

Although the previous section developed general relations for small deformation theory, we now wish to establish a more geometrical interpretation of these results. Typically, elasticity variables and equations are field quantities defined at each point in the material continuum. However, particular field equations are often developed by first investigating the behavior of infinitesimal elements (with coordinate boundaries), and then a limiting process is invoked that allows the element to shrink to a point. Thus, consider the common deformational behavior of a rectangular element as shown in Figure 2-3. The usual types of motion include rigid-body rotation and extensional and shearing deformations as illustrated. Rigid-body motion does not contribute to the strain field, and thus also does not affect the stresses. We therefore focus our study primarily on the extensional and shearing deformation.

Figure 2-4 illustrates the two-dimensional deformation of a rectangular element with original dimensions $dx$ by $dy$. After deformation, the element takes a rhombus form as shown in the dotted outline. The displacements of various corner reference points are indicated

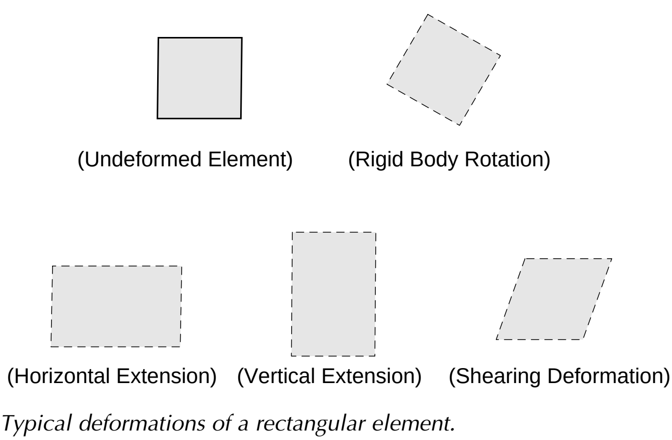



**FIGURE 2-3** *Typical deformations of a rectangular element.*

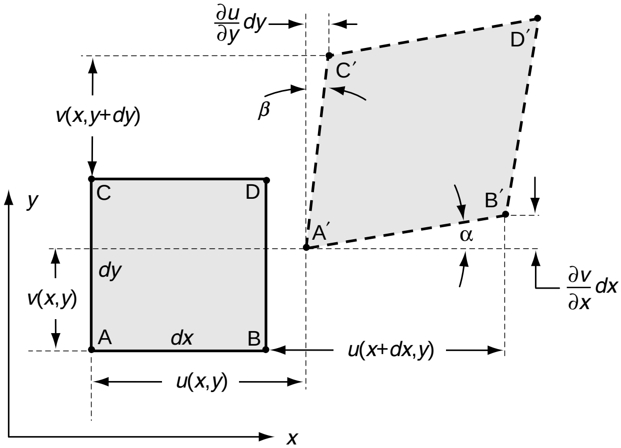



**FIGURE 2-4** *Two-dimensional geometric strain deformation.*

in the figure. Reference point $A$ is taken at location $(x,y)$, and the displacement components of this point are thus $u(x,y)$ and $v(x,y)$. The corresponding displacements of point $B$ are $u(x+dx, y)$ and $v(x+dx, y)$, and the displacements of the other corner points are defined in an analogous manner. According to small deformation theory, $u(x+dx, y) \approx u(x, y)+ (\partial u/\partial x)\, dx$, with similar expansions for all other terms.

The *normal* or *extensional strain component* in a direction $n$ is defined as the change in length per unit length of fibers oriented in the $n$-direction. Normal strain is positive if fibers increase in length and negative if the fiber is shortened. In Figure 2-4, the normal strain in the $x$ direction can thus be defined by

$$\varepsilon_x = \frac{A'B' - AB}{AB}$$

From the geometry in Figure 2-4,

$$A'B' = \sqrt{\left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\, dx \approx \left(1 + \frac{\partial u}{\partial x}\right)dx$$

where, consistent with small deformation theory, we have dropped the higher-order terms. Using these results and the fact that $AB = dx$, the normal strain in the $x$-direction reduces to

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{2.2.1}$$

In similar fashion, the normal strain in the $y$-direction becomes

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{2.2.2}$$

A second type of strain is shearing deformation, which involves angles changes (see Figure 2-3). *Shear strain* is defined as the change in angle between two originally orthogonal directions in the continuum material. This definition is actually referred to as the *engineering shear strain*. Theory of elasticity applications generally use a tensor formalism that requires a shear strain definition corresponding to one-half the angle change between orthogonal axes; see previous relation $(2.1.7)_1$. Measured in radians, shear strain is positive if the right angle between the positive directions of the two axes decreases. Thus, the sign of the shear strain depends on the coordinate system. In Figure 2-4, the engineering shear strain with respect to the $x$- and $y$-directions can be defined as

$$\gamma_{xy} = \frac{\pi}{2} - \angle C'A'B' = \alpha + \beta$$

For small deformations, $\alpha \approx \tan\alpha$ and $\beta \approx \tan\beta$, and the shear strain can then be expressed as

$$\gamma_{xy} = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} + \frac{\frac{\partial u}{\partial y}dy}{dy + \frac{\partial v}{\partial y}dy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.2.3}$$

where we have again neglected higher-order terms in the displacement gradients. Note that each derivative term is positive if lines $AB$ and $AC$ rotate inward as shown in the figure. By simple interchange of $x$ and $y$ and $u$ and $v$, it is apparent that $\gamma_{xy} = \gamma_{yx}$.

By considering similar behaviors in the $y$-$z$ and $x$-$z$ planes, these results can be easily extended to the general three-dimensional case, giving the results:

$$\begin{gathered} \varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{gathered} \tag{2.2.4}$$

Thus, we define three normal and three shearing strain components leading to a total of six independent components that completely describe small deformation theory. This set of equations is normally referred to as the *strain-displacement relations*. However, these results are written in terms of the engineering strain components, and tensorial elasticity theory prefers to use the strain tensor $e_{ij}$ defined by $(2.1.7)_1$. This represents only a minor change because the normal strains are identical and shearing strains differ by a factor of one-half; for example, $e_{11} = e_x = \varepsilon_x$ and $e_{12} = e_{xy} = 1/2\gamma_{xy}$, and so forth.

Therefore, using the strain tensor $e_{ij}$, the strain-displacement relations can be expressed in component form as

$$\begin{gathered} e_x = \frac{\partial u}{\partial x}, e_y = \frac{\partial v}{\partial y},\ e_z = \frac{\partial w}{\partial z} \\ e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\ e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{gathered} \tag{2.2.5}$$

Using the more compact tensor notation, these relations are written as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.2.6}$$

while in direct vector/matrix notation as the form reads:

$$\boldsymbol{e} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right] \tag{2.2.7}$$

where $\boldsymbol{e}$ is the strain matrix and $\nabla \boldsymbol{u}$ is the displacement gradient matrix and $(\nabla \boldsymbol{u})^T$ is its transpose.

The strain is a symmetric second-order tensor ($e_{ij} = e_{ji}$) and is commonly written in matrix format:

$$\boldsymbol{e} = [\boldsymbol{e}] = \begin{bmatrix} e_x & e_{xy} & e_{xz} \\ e_{xy} & e_y & e_{yz} \\ e_{xz} & e_{yz} & e_z \end{bmatrix} \tag{2.2.8}$$

Before we conclude this geometric presentation, consider the rigid-body rotation of our two-dimensional element in the $x$-$y$ plane, as shown in Figure 2-5. If the element is rotated through a small rigid-body angular displacement about the $z$-axis, using the bottom element edge, the rotation angle is determined as $\partial v/\partial x$, while using the left edge, the angle is given by $-\partial u/\partial y$. These two expressions are of course the same; that is, $\partial v/\partial x = -\partial u/\partial y$ and note that this would imply $e_{xy} = 0$. The rotation can then be expressed as $\omega_z = [(\partial v/\partial x) - (\partial u/\partial y)]/2$, which matches with the expression given earlier in $(2.1.9)_3$. The other components of rotation follow in an analogous manner.

Relations for the constant rotation $\omega_z$ can be integrated to give the result:

$$\begin{aligned} u^* &= u_o - \omega_z y \\ v^* &= v_o + \omega_z x \end{aligned} \tag{2.2.9}$$

where $u_o$ and $v_o$ are arbitrary constant translations in the $x$- and $y$-directions. This result then specifies the general form of the displacement field for two-dimensional rigid-body motion. We can easily verify that the displacement field given by (2.2.9) yields zero strain.

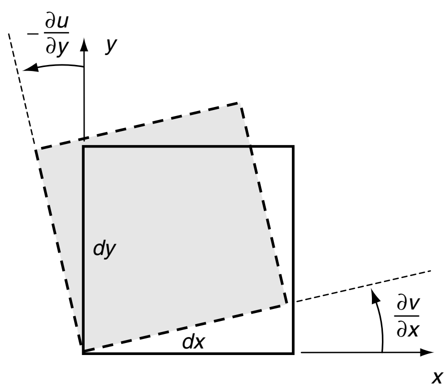



**FIGURE 2-5** *Two-dimensional rigid-body rotation.*

For the three-dimensional case, the most general form of rigid-body displacement can be expressed as

$$\begin{aligned}u^* &= u_o - \omega_z y + \omega_y z \\ v^* &= v_o - \omega_x z + \omega_z x \\ w^* &= w_o - \omega_y x + \omega_x y\end{aligned} \tag{2.2.10}$$

As shown later, integrating the strain-displacement relations to determine the displacement field produces arbitrary constants and functions of integration, which are equivalent to rigid-body motion terms of the form given by (2.2.9) or (2.2.10). Thus, it is important to recognize such terms because we normally want to drop them from the analysis since they do not contribute to the strain or stress fields.

## 2.3 Strain Transformation

Because the strains are components of a second-order tensor, the transformation theory discussed in Section 1.5 can be applied. Transformation relation $(1.5.1)_3$ is applicable for second-order tensors, and applying this to the strain gives

$$e'_{ij} = Q_{ip}Q_{jq}e_{pq} \tag{2.3.1}$$

where the rotation matrix $Q_{ij} = \cos(x'_i,\ x_j)$. Thus, given the strain in one coordinate system, we can determine the new components in any other rotated system. For the general three-dimensional case, define the rotation matrix as

$$Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{2.3.2}$$

Using this notational scheme, the specific transformation relations from equation (2.3.1) become

$$\begin{aligned}
e'_x &= e_x l_1^2 + e_y m_1^2 + e_z n_1^2 + 2(e_{xy} l_1 m_1 + e_{yz} m_1 n_1 + e_{zx} n_1 l_1) \\
e'_y &= e_x l_2^2 + e_y m_2^2 + e_z n_2^2 + 2(e_{xy} l_2 m_2 + e_{yz} m_2 n_2 + e_{zx} n_2 l_2) \\
e'_z &= e_x l_3^2 + e_y m_3^2 + e_z n_3^2 + 2(e_{xy} l_3 m_3 + e_{yz} m_3 n_3 + e_{zx} n_3 l_3) \\
e'_{xy} &= e_x l_1 l_2 + e_y m_1 m_2 + e_z n_1 n_2 + e_{xy}(l_1 m_2 + m_1 l_2) + e_{yz}(m_1 n_2 + n_1 m_2) + e_{zx}(n_1 l_2 + l_1 n_2) \\
e'_{yz} &= e_x l_2 l_3 + e_y m_2 m_3 + e_z n_2 n_3 + e_{xy}(l_2 m_3 + m_2 l_3) + e_{yz}(m_2 n_3 + n_2 m_3) + e_{zx}(n_2 l_3 + l_2 n_3) \\
e'_{zx} &= e_x l_3 l_1 + e_y m_3 m_1 + e_z n_3 n_1 + e_{xy}(l_3 m_1 + m_3 l_1) + e_{yz}(m_3 n_1 + n_3 m_1) + e_{zx}(n_3 l_1 + l_3 n_1)
\end{aligned} \tag{2.3.3}$$

For the two-dimensional case shown in Figure 2-6, the transformation matrix can be expressed as

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.3.4}$$

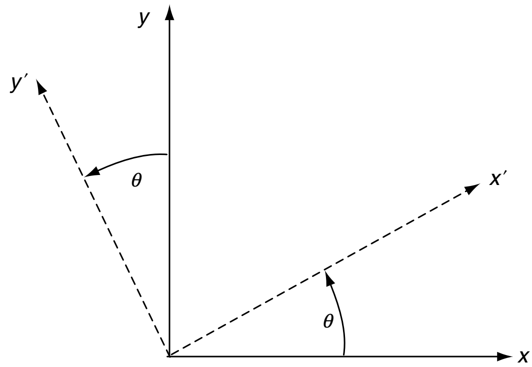



**FIGURE 2-6** *Two-dimensional rotational transformation.*

Under this transformation, the in-plane strain components transform according to

$$
\begin{aligned}
e'_x &= e_x \cos^2\theta + e_y \sin^2\theta + 2e_{xy}\sin\theta\cos\theta \\
e'_y &= e_x \sin^2\theta + e_y \cos^2\theta - 2e_{xy}\sin\theta\cos\theta \\
e'_{xy} &= -e_x \sin\theta\cos\theta + e_y \sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned}
\tag{2.3.5}
$$

which is commonly rewritten in terms of the double angle:

$$
\begin{aligned}
e'_x &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta \\
e'_y &= \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta \\
e'_{xy} &= \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta
\end{aligned}
\tag{2.3.6}
$$

Transformation relations (2.3.6) can be directly applied to establish transformations between Cartesian and polar coordinate systems (see Exercise 2-6). Additional applications of these results can be found when dealing with experimental strain gage measurement systems. For example, standard experimental methods using a rosette strain gage allow the determination of extensional strains in three different directions on the surface of a structure. Using this type of data, relation $(2.3.6)_1$ can be repeatedly used to establish three independent equations that can be solved for the state of strain $(e_x, e_y, e_{xy})$ at the surface point under study (see Exercise 2-7).

Both two- and three-dimensional transformation equations can be easily incorporated in MATLAB to provide numerical solutions to problems of interest. Such examples are given in Exercises 2-8 and 2-9.

## 2.4 Principal Strains

From the previous discussion in Section 1.6, it follows that because the strain is a symmetric second-order tensor, we can identify and determine its principal axes and values. According to this theory, for any given strain tensor we can establish the principal value problem and solve the characteristic equation to explicitly determine the principal values and directions. The general characteristic equation for the strain tensor can be written as

$$\det[e_{ij} - e\delta_{ij}] = -e^3 + \vartheta_1 e^2 - \vartheta_2 e + \vartheta_3 = 0 \tag{2.4.1}$$

where $e$ is the *principal strain* and the fundamental invariants of the strain tensor can be expressed in terms of the three principal strains $e_1, e_2, e_3$ as

$$\begin{aligned} \vartheta_1 &= e_1 + e_2 + e_3 \\ \vartheta_2 &= e_1 e_2 + e_2 e_3 + e_3 e_1 \\ \vartheta_3 &= e_1 e_2 e_3 \end{aligned} \tag{2.4.2}$$

The first invariant $\vartheta_1 = \vartheta$ is normally called the *cubical dilatation,* because it is related to the change in volume of material elements (see Exercise 2-11).

The strain matrix in the principal coordinate system takes the special diagonal form

$$e_{ij} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} \tag{2.4.3}$$

Notice that for this principal coordinate system, the deformation does not produce any shearing and thus is only extensional. Therefore, a rectangular element oriented along principal axes of strain will retain its orthogonal shape and undergo only extensional deformation of its sides.

## 2.5 Spherical and Deviatoric Strains

In particular applications it is convenient to decompose the strain tensor into two parts called *spherical* and *deviatoric strain tensors*. The spherical strain is defined by

$$\tilde{e}_{ij} = \frac{1}{3} e_{kk} \delta_{ij} = \frac{1}{3} \vartheta \delta_{ij} \tag{2.5.1}$$

while the deviatoric strain is specified as

$$\hat{e}_{ij} = e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \tag{2.5.2}$$

Note that the total strain is then simply the sum

$$e_{ij} = \tilde{e}_{ij} + \hat{e}_{ij} \tag{2.5.3}$$

The spherical strain represents only volumetric deformation and is an isotropic tensor, being the same in all coordinate systems (as per the discussion in Section 1.5). The deviatoric strain tensor then accounts for changes in shape of material elements. It can be shown that the principal directions of the deviatoric strain are the same as those of the strain tensor.

# Appendix

# B

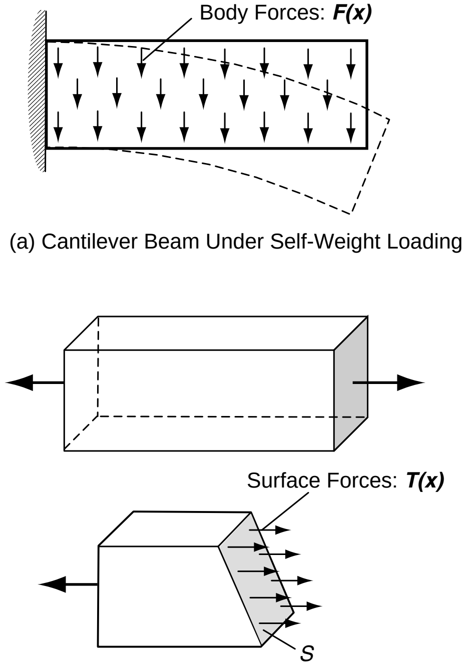
# 3 Stress and Equilibrium

The previous chapter investigated the kinematics of deformation without regard to the force or stress distribution within the elastic solid. We now wish to examine these issues and explore the transmission of forces through deformable materials. Our study leads to the definition and use of the traction vector and stress tensor. Each provides a quantitative method to describe both boundary and internal force distributions within a continuum solid. Because it is commonly accepted that maximum stresses are a major contributing factor to material failure, primary application of elasticity theory is used to determine the distribution of stress within a given structure. Related to these force distribution issues is the concept of equilibrium. Within a deformable solid, the force distribution at each point must be balanced. For the static case, the summation of forces on an infinitesimal element is required to be zero, while for a dynamic problem the resultant force must equal the mass times the element's acceleration. In this chapter, we establish the definitions and properties of the traction vector and stress tensor and develop the equilibrium equations, which become another set of field equations necessary in the overall formulation of elasticity theory. It should be noted that the developments in this chapter do not require that the material be elastic, and thus in principle these results apply to a broader class of material behavior.

## 3.1 Body and Surface Forces

When a structure is subjected to applied external loadings, internal forces are induced inside the body. Following the philosophy of continuum mechanics, these internal forces are distributed continuously within the solid. In order to study such forces, it is convenient to categorize them into two major groups, commonly referred to as body forces and surface forces.

*Body forces* are proportional to the body's mass and are reacted with an agent outside of the body. Examples of these include gravitational-weight forces, magnetic forces, and inertial forces. Figure 3-1(a) shows an example body force of an object's self-weight. By using continuum mechanics principles, a *body force density* (force per unit volume) $\boldsymbol{F}(\boldsymbol{x})$ can be defined such that the total resultant body force of an entire solid can be written as a volume integral over the body




(a) Cantilever Beam Under Self-Weight Loading

(b) Sectioned Axially Loaded Beam

**FIGURE 3-1** *Examples of body and surface forces.*

$$\boldsymbol{F_R} = \iiint_V \boldsymbol{F}(\boldsymbol{x})dV \tag{3.1.1}$$

*Surface forces* always act on a surface and result from physical contact with another body. Figure 3-1(b) illustrates surface forces existing in a beam section that has been created by sectioning the body into two pieces. For this particular case, the surface $S$ is a virtual one in the sense that it was artificially created to investigate the nature of the internal forces at this location in the body. Again the resultant surface force over the entire surface $S$ can be expressed as the integral of a *surface force density function* $\boldsymbol{T}^n(\boldsymbol{x})$

$$\boldsymbol{F}_S = \iint_S \boldsymbol{T}^{\boldsymbol{n}}(\boldsymbol{x})dS \tag{3.1.2}$$

The surface force density is normally referred to as the traction vector and is discussed in more detail in the next section. In the development of classical elasticity, distributions of body or surface couples are normally not included. Theories that consider such force distributions have been constructed in an effort to extend classical elasticity for applications in micromechanical modeling. Such approaches are normally called *micropolar* or *couple-stress theory* (see Eringen 1968) and are briefly presented in Chapter 14.

## 3.2 Traction Vector and Stress Tensor

In order to quantify the nature of the internal distribution of forces within a continuum solid, consider a general body subject to arbitrary (concentrated and distributed) external loadings, as shown in Figure 3-2. To investigate the internal forces, a section is made through the body as shown. On this section consider a small area $\Delta A$ with unit normal vector $\boldsymbol{n}$. The resultant surface force acting on $\Delta A$ is defined by $\Delta \boldsymbol{F}$. Consistent with our earlier discussion, no resultant surface couple is included. The *stress* or *traction vector* is defined by

$$\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n}) = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A} \tag{3.2.1}$$

Notice that the traction vector depends on both the spatial location and the unit normal vector to the surface under study. Thus, even though we may be investigating the same point, the traction vector still varies as a function of the orientation of the surface normal. Because the traction is defined as force per unit area, the total surface force is determined through integration as per relation (3.1.2). Note, also, the simple action-reaction principle (Newton's third law)

$$\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{T}^n(\boldsymbol{x}, -\boldsymbol{n})$$

Consider now the special case in which $\Delta A$ coincides with each of the three coordinate planes with the unit normal vectors pointing along the positive coordinate axes. This concept is shown in Figure 3-3, where the three coordinate surfaces for $\Delta A$ partition off a cube of material. For this case, the traction vector on each face can be written as

$$\begin{aligned}
\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_1) &= \sigma_x \boldsymbol{e}_1 + \tau_{xy} \boldsymbol{e}_2 + \tau_{xz} \boldsymbol{e}_3 \\
\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_2) &= \tau_{yx} \boldsymbol{e}_1 + \sigma_y \boldsymbol{e}_2 + \tau_{yz} \boldsymbol{e}_3 \\
\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_3) &= \tau_{zx} \boldsymbol{e}_1 + \tau_{zy} \boldsymbol{e}_2 + \sigma_z \boldsymbol{e}_3
\end{aligned} \tag{3.2.2}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the unit vectors along each coordinate direction, and the nine quantities $\{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zy}, \tau_{zx}, \tau_{xz}\}$ are the components of the traction vector on each of three coordinate planes as illustrated. These nine components are called the *stress components,*

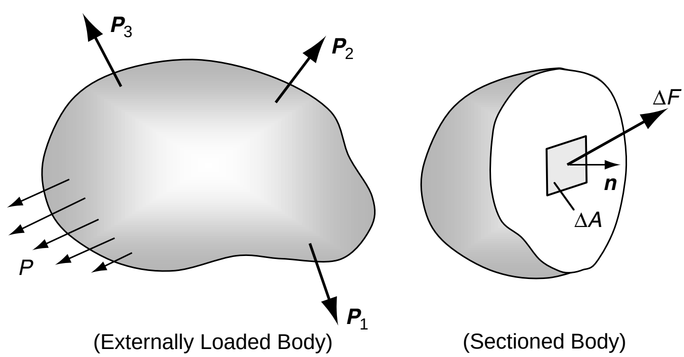



**FIGURE 3-2** *Sectioned solid under external loading.*

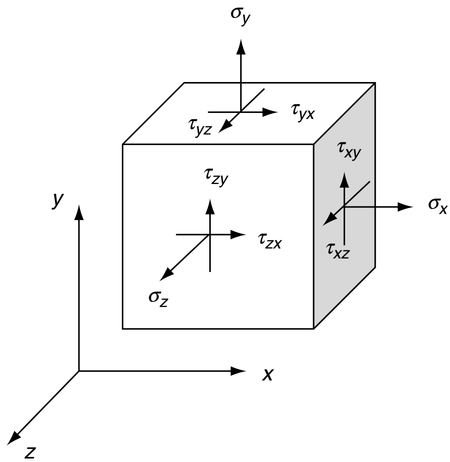



**FIGURE 3-3** *Components of the stress.*

with $\sigma_x,\ \sigma_y,\ \sigma_z$ referred to as *normal stresses* and $\tau_{xy},\ \tau_{yx},\ \tau_{yz},\ \tau_{zy},\ \tau_{zx},\ \tau_{xz}$ called the *shearing stresses*. The components of stress $\sigma_{ij}$ are commonly written in matrix format

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{3.2.3}$$

and it can be formally shown that the stress is a second-order tensor that obeys the appropriate transformation law $(1.5.3)_3$.

The positive directions of each stress component are illustrated in Figure 3-3. Regardless of the coordinate system, positive normal stress always acts in tension out of the face, and only one subscript is necessary because it always acts normal to the surface. The shear stress, however, requires two subscripts, the first representing the plane of action and the second designating the direction of the stress. Similar to shear strain, the sign of the shear stress depends on coordinate system orientation. For example, on a plane with a normal in the positive $x$ direction, positive $\tau_{xy}$ acts in the positive $y$ direction. Similar definitions follow for the other shear stress components. In subsequent chapters, proper formulation of elasticity problems requires knowledge of these basic definitions, directions, and sign conventions for particular stress components.

Consider next the traction vector on an oblique plane with arbitrary orientation, as shown in Figure 3-4. The unit normal to the surface can be expressed by

$$\boldsymbol{n} = n_x \boldsymbol{e}_1 + n_y \boldsymbol{e}_2 + n_z \boldsymbol{e}_3 \tag{3.2.3}$$

where $n_x,\ n_y,\ n_z$ are the direction cosines of the unit vector $\boldsymbol{n}$ relative to the given coordinate system. We now consider the equilibrium of the pyramidal element interior to the oblique and coordinate planes. Invoking the force balance between tractions on the oblique and coordinate faces gives

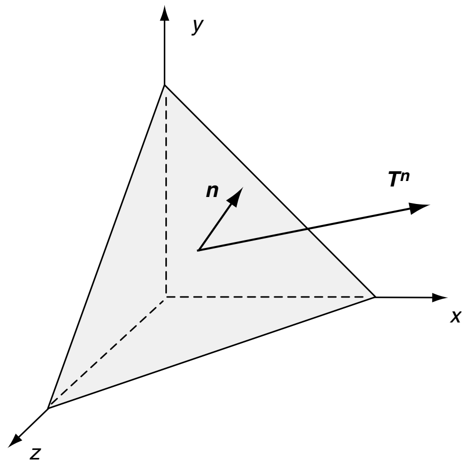



**FIGURE 3-4** *Traction on an oblique plane.*

$$\boldsymbol{T}^n = n_x\boldsymbol{T}^n(\boldsymbol{n}=\boldsymbol{e}_1) + n_y\boldsymbol{T}^n(\boldsymbol{n}=\boldsymbol{e}_2) + n_z\boldsymbol{T}^n(\boldsymbol{n}=\boldsymbol{e}_3)$$

and by using relations (3.2.2), this can be written as

$$\begin{aligned}\boldsymbol{T}^{\boldsymbol{n}} &= (\sigma_x n_x + \tau_{yx}n_y + \tau_{zx}n_z)\boldsymbol{e}_1 \\ &\quad + (\tau_{xy}n_x + \sigma_y n_y + \tau_{zy}n_z)\boldsymbol{e}_2 \\ &\quad + (\tau_{xz}n_x + \tau_{yz}n_y + \sigma_z n_z)\boldsymbol{e}_3\end{aligned} \tag{3.2.4}$$

or in index notation

$$T_i^n = \sigma_{ji}n_j \tag{3.2.5}$$

Relation (3.2.4) or (3.2.5) provides a simple and direct method to calculate the forces on oblique planes and surfaces. This technique proves to be very useful to specify general boundary conditions during the formulation and solution of elasticity problems.

Following the principles of small deformation theory, the previous definitions for the stress tensor and traction vector do not make a distinction between the deformed and undeformed configurations of the body. As mentioned in the previous chapter, such a distinction only leads to small modifications that are considered *higher-order* effects and are normally neglected. However, for large deformation theory, sizeable differences exist between these configurations, and the undeformed configuration (commonly called the reference configuration) is often used in problem formulation. This gives rise to the definition of an additional stress called the *Piola-Kirchhoff stress tensor* that represents the force per unit area in the reference configuration (see Chandrasekharaiah and Debnath 1994). In the more general scheme, the stress $\sigma_{ij}$ is referred to as the *Cauchy* stress tensor. Throughout the text only small deformation theory is considered, and thus the distinction between these two definitions of stress disappears, thereby eliminating any need for this additional terminology.

## 3.3 Stress Transformation

Analogous to our previous discussion with the strain tensor, the stress components must also follow the standard transformation rules for second-order tensors established in Section 1.5. Applying transformation relation $(1.5.1)_3$ for the stress gives

$$\sigma'_{ij} = Q_{ip}Q_{jq}\sigma_{pq} \tag{3.3.1}$$

where the rotation matrix $Q_{ij} = \cos(x'_i, x_j)$. Therefore, given the stress in one coordinate system, we can determine the new components in any other rotated system. For the general three-dimensional case, the rotation matrix may be chosen in the form

$$Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{3.3.2}$$

Using this notational scheme, the specific transformation relations for the stress then become

$$\begin{aligned}
\sigma'_x &= \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1) \\
\sigma'_y &= \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2(\tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{zx} n_2 l_2) \\
\sigma'_z &= \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2(\tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{zx} n_3 l_3) \\
\tau'_{xy} &= \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + m_1 l_2) + \tau_{yz}(m_1 n_2 + n_1 m_2) + \tau_{zx}(n_1 l_2 + l_1 n_2) \\
\tau'_{yz} &= \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}(l_2 m_3 + m_2 l_3) + \tau_{yz}(m_2 n_3 + n_2 m_3) + \tau_{zx}(n_2 l_3 + l_2 n_3) \\
\tau'_{zx} &= \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}(l_3 m_1 + m_3 l_1) + \tau_{yz}(m_3 n_1 + n_3 m_1) + \tau_{zx}(n_3 l_1 + l_3 n_1)
\end{aligned} \tag{3.3.3}$$

For the two-dimensional case originally shown in Figure 2-6, the transformation matrix was given by relation (2.3.4). Under this transformation, the in-plane stress components transform according to

$$\begin{aligned}
\sigma'_x &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\
\sigma'_y &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \\
\tau'_{xy} &= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned} \tag{3.3.4}$$

which is commonly rewritten in terms of the double angle

$$\begin{aligned}
\sigma'_x &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\
\sigma'_y &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\
\tau'_{xy} &= \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta
\end{aligned} \tag{3.3.5}$$

Similar to our discussion on strain in the previous chapter, relations (3.3.5) can be directly applied to establish stress transformations between Cartesian and polar coordinate systems (see Exercise 3-3). Both two- and three-dimensional stress transformation equations can be easily incorporated in MATLAB to provide numerical solution to problems of interest (see Exercise 3-2).

## 3.4 Principal Stresses

We can again use the previous developments from Section 1.6 to discuss the issues of principal stresses and directions. It is shown later in the chapter that the stress is a symmetric tensor. Using this fact, appropriate theory has been developed to identify and determine principal axes and values for the stress. For any given stress tensor we can establish the principal value problem and solve the characteristic equation to explicitly determine the principal values and directions. The general characteristic equation for the stress tensor becomes

$$\det[\sigma_{ij} - \sigma\delta_{ij}] = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0 \tag{3.4.1}$$

where $\sigma$ are the *principal stresses* and the fundamental invariants of the stress tensor can be expressed in terms of the three principal stresses $\sigma_1, \sigma_2, \sigma_3$ as

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{3.4.2}$$

In the principal coordinate system, the stress matrix takes the special diagonal form

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{3.4.3}$$

A comparison of the general and principal stress states is shown in Figure 3-5. Notice that for the principal coordinate system, all shearing stresses vanish and thus the state includes only normal stresses. These issues should be compared to the equivalent comments made for the strain tensor at the end of Section 2.4.

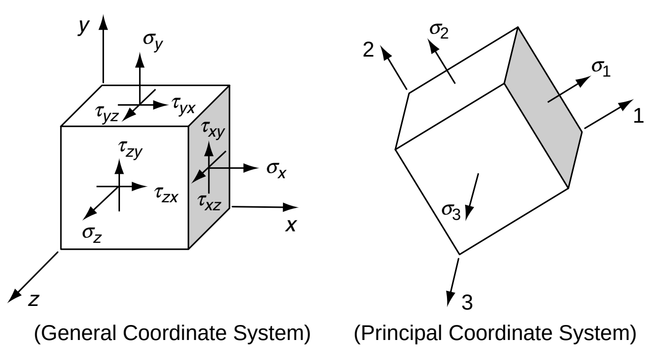



**FIGURE 3-5** *Comparison of general and principal stress states.*

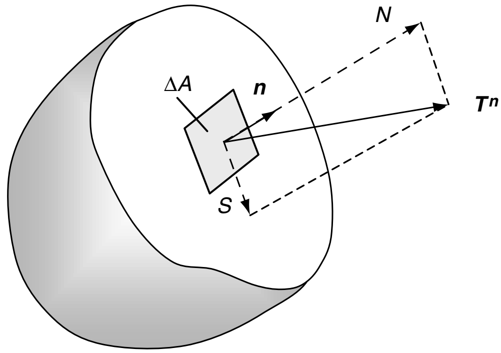



**FIGURE 3-6** *Traction vector decomposition.*

We now wish to go back to investigate another issue related to stress and traction transformation that makes use of principal stresses. Consider the general traction vector $\boldsymbol{T}^n$ that acts on an arbitrary surface as shown in Figure 3-6. The issue of interest is to determine the traction vector's normal and shear components $N$ and $S$. The normal component is simply the traction's projection in the direction of the unit normal vector $\boldsymbol{n}$, while the shear component is found by Pythagorean theorem,

$$\begin{aligned} N &= \boldsymbol{T}^n \cdot \boldsymbol{n} \\ S &= (|\boldsymbol{T}^n|^2 - N^2)^{1/2} \end{aligned} \tag{3.4.4}$$

Using the relationship for the traction vector (3.2.5) into $(3.4.4)_1$ gives

$$\begin{aligned} N &= \boldsymbol{T}^n \cdot \boldsymbol{n} = T_i^n n_i = \sigma_{ji} n_j n_i \\ &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \end{aligned} \tag{3.4.5}$$

where, in order to simplify the expressions, we have used the principal axes for the stress tensor. In a similar manner,

$$\begin{aligned} |\boldsymbol{T}^n|^2 &= \boldsymbol{T}^n \cdot \boldsymbol{T}^n = T_i^n T_i^n = \sigma_{ji} n_j \sigma_{ki} n_k \\ &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \end{aligned} \tag{3.4.6}$$

Using these results back into relation (3.4.4) yields

$$\begin{aligned} N &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \\ S^2 + N^2 &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \end{aligned} \tag{3.4.7}$$

In addition, we also add the condition that the vector $\boldsymbol{n}$ has unit magnitude

$$1 = n_1^2 + n_2^2 + n_3^2 \tag{3.4.8}$$

Relations (3.4.7) and (3.4.8) can be viewed as three linear algebraic equations for the unknowns $n_1^2, n_2^2, n_3^2$. Solving this system gives the following result:

$$n_1^2 = \frac{S^2 + (N - \sigma_2)(N - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$
$$n_2^2 = \frac{S^2 + (N - \sigma_3)(N - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \qquad (3.4.9)$$
$$n_3^2 = \frac{S^2 + (N - \sigma_1)(N - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$

Without loss in generality, we can rank the principal stresses as $\sigma_1 > \sigma_2 > \sigma_3$. Noting that the expressions given by (3.4.9) must be greater than or equal to zero, we can conclude the following

$$S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0$$
$$S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0 \qquad (3.4.10)$$
$$S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0$$

For the equality case, equations (3.4.10) represent three circles in an *S-N* coordinate system, and Figure 3-7 illustrates the location of each circle. These results were originally generated by Otto Mohr over a century ago, and the circles are commonly called *Mohr's circles of stress*. The three inequalities given in (3.4.10) imply that all admissible values of *N* and *S* lie in the shaded regions bounded by the three circles. Note that, for the ranked principal stresses, the largest shear component is easily determined as $S_{max} = 1/2|\sigma_1 - \sigma_3|$. Although these circles can be effectively used for two-dimensional stress transformation, the general tensorial-based equations (3.3.3) are normally used for general transformation computations.

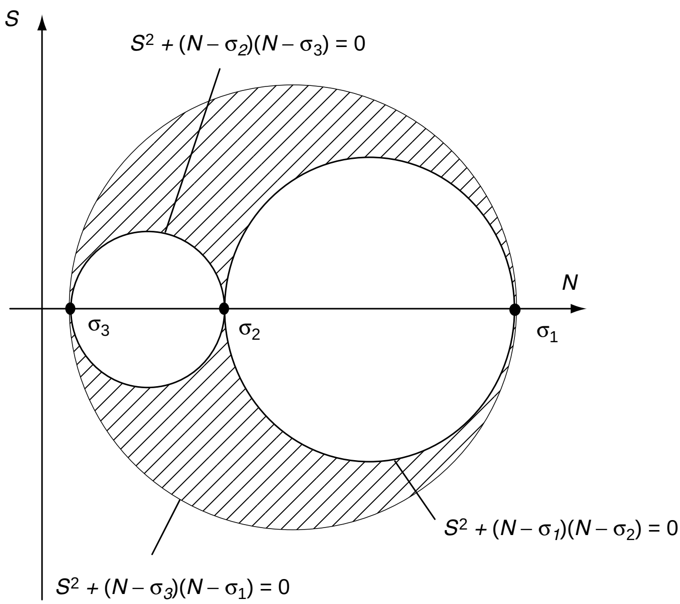



**FIGURE 3-7** *Mohr's circles of stress.*

### EXAMPLE 3-1: Stress Transformation

For the following state of stress, determine the principal stresses and directions and find the traction vector on a plane with unit normal $\boldsymbol{n} = (0, 1, 1)/\sqrt{2}$.

$$\sigma_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

The principal stress problem is started by calculating the three invariants, giving the result $I_1 = 3,\ I_2 = -6,\ I_3 = -8$. This yields the following characteristic equation:

$$-\sigma^3 + 3\sigma^2 + 6\sigma - 8 = 0$$

The roots of this equation are found to be $\sigma = 4,\ 1,\ -2$. Back-substituting the first root into the fundamental system (see 1.6.1) gives

$$\begin{aligned} -n_1^{(1)} + n_2^{(1)} + n_3^{(1)} &= 0 \\ n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} &= 0 \\ n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} &= 0 \end{aligned}$$

Solving this system, the normalized principal direction is found to be $\boldsymbol{n}^{(1)} = (2,\ 1,\ 1)/\sqrt{6}$. In similar fashion the other two principal directions are $\boldsymbol{n}^{(2)} = (-1,\ 1,\ 1)/\sqrt{3}$, $\boldsymbol{n}^{(3)} = (0,\ -1,\ 1)/\sqrt{2}$.

The traction vector on the specified plane is calculated by using the relation

$$T_i^n = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

## 3.5 Spherical and Deviatoric Stresses

As mentioned in our previous discussion on strain, it is often convenient to decompose the stress into two parts called the *spherical* and *deviatoric stress tensors*. Analogous to relations (2.5.1) and (2.5.2), the spherical stress is defined by

$$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{3.5.1}$$

while the deviatoric stress becomes

$$\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{3.5.2}$$

Note that the total stress is then simply the sum

$$\sigma_{ij} = \tilde{\sigma}_{ij} + \hat{\sigma}_{ij} \tag{3.5.3}$$

The spherical stress is an isotropic tensor, being the same in all coordinate systems (as per discussion in Section 1.5). It can be shown that the principal directions of the deviatoric stress are the same as those of the stress tensor (see Exercise 3-8).

## 3.6 Equilibrium Equations

The stress field in an elastic solid is continuously distributed within the body and uniquely determined from the applied loadings. Because we are dealing primarily with bodies in equilibrium, the applied loadings satisfy the equations of static equilibrium; that is, the summation of forces and moments is zero. If the entire body is in equilibrium, then all parts must also be in equilibrium. Thus, we can partition any solid into an appropriate subdomain and apply the equilibrium principle to that region. Following this approach, equilibrium equations can be developed that express the vanishing of the resultant force and moment at a continuum point in the material. These equations can be developed by using either an arbitrary finite subdomain or a special differential region with boundaries coinciding with coordinate surfaces. We shall formally use the first method in the text, and the second scheme is included in Exercises 3-10 and 3-11.

Consider a closed subdomain with volume $V$ and surface $S$ within a body in equilibrium. The region has a general distribution of surface tractions $\boldsymbol{T}^n$ body forces $\boldsymbol{F}$ as shown in Figure 3-8. For static equilibrium, conservation of linear momentum implies that the forces acting on this region are balanced and thus the resultant force must vanish. This concept can be easily written in index notation as

$$\iint_S T_i^n dS + \iiint_V F_i dV = 0 \tag{3.6.1}$$

Using relation (3.2.5) for the traction vector, we can express the equilibrium statement in terms of stress:

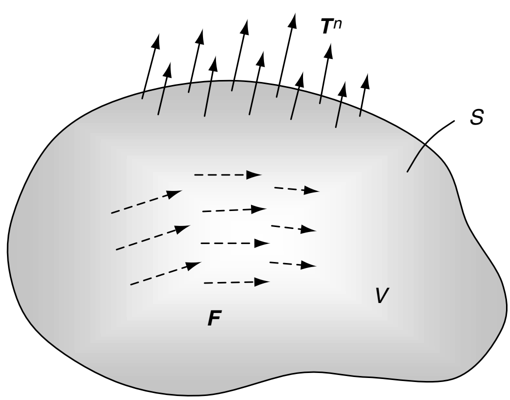



**FIGURE 3-8** *Body and surface forces acting on arbitrary portion of a continuum.*

$$\iint_S \sigma_{ji} n_j dS + \iiint_V F_i dV = 0 \tag{3.6.2}$$

Applying the divergence theorem (1.8.7) to the surface integral allows the conversion to a volume integral, and relation (3.6.2) can then be expressed as

$$\iiint_V (\sigma_{ji,j} + F_i) dV = 0 \tag{3.6.3}$$

Because the region $V$ is arbitrary (any part of the medium can be chosen) and the integrand in (3.6.3) is continuous, then by the zero-value theorem (1.8.12), the integrand must vanish:

$$\sigma_{ji,j} + F_i = 0 \tag{3.6.4}$$

This result represents three scalar relations called the *equilibrium equations*. Written in scalar notation they are

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0
\end{aligned} \tag{3.6.5}$$

Thus, all elasticity stress fields must satisfy these relations in order to be in static equilibrium.

Next consider the angular momentum principle that states that the moment of all forces acting on any portion of the body must vanish. Note that the point about which the moment is calculated can be chosen arbitrarily. Applying this principle to the region shown in Figure 3-8 results in a statement of the vanishing of the moments resulting from surface and body forces:

$$\iint_S \varepsilon_{ijk} x_j T_k^n dS + \iiint_V \varepsilon_{ijk} x_j F_k dV = 0 \tag{3.6.6}$$

Again using relation (3.2.5) for the traction, (3.6.6) can be written as

$$\iint_S \varepsilon_{ijk} x_j \sigma_{lk} n_l dS + \iiint_V \varepsilon_{ijk} x_j F_k dV = 0$$

and application of the divergence theorem gives

$$\iiint_V [(\varepsilon_{ijk} x_j \sigma_{lk})_{,l} + \varepsilon_{ijk} x_j F_k] dV = 0$$

This integral can be expanded and simplified as

$$\iiint_V [\varepsilon_{ijk}x_{j,l}\sigma_{lk} + \varepsilon_{ijk}x_j\sigma_{lk,l} + \varepsilon_{ijk}x_jF_k]dV =$$
$$\iiint_V [\varepsilon_{ijk}\delta_{jl}\sigma_{lk} + \varepsilon_{ijk}x_j\sigma_{lk,l} + \varepsilon_{ijk}x_jF_k]dV =$$
$$\iiint_V [\varepsilon_{ijk}\sigma_{jk} - \varepsilon_{ijk}x_jF_k + \varepsilon_{ijk}x_jF_k]dV = \iiint_V \varepsilon_{ijk}\sigma_{jk}dV$$

where we have used the equilibrium equations (3.6.4) to simplify the final result. Thus, (3.6.6) now gives

$$\iiint_V \varepsilon_{ijk}\sigma_{jk}dV = 0$$

As per our earlier arguments, because the region $V$ is arbitrary, the integrand must vanish, giving $\varepsilon_{ijk}\sigma_{jk} = 0$. However, because the alternating symbol is antisymmetric in indices $jk$, the other product term $\sigma_{jk}$ must be symmetric, thus implying

$$\begin{aligned} &\tau_{xy} = \tau_{yx} \\ \sigma_{ij} = \sigma_{ji} \Rightarrow\ &\tau_{yz} = \tau_{zy} \\ &\tau_{zx} = \tau_{xz} \end{aligned} \tag{3.6.7}$$

We thus find that, similar to the strain, the stress tensor is also symmetric and therefore has only six independent components in three dimensions. Under these conditions, the equilibrium equations can then be written as

$$\sigma_{ij,j} + F_i = 0 \tag{3.6.8}$$

## 3.7 Relations in Curvilinear Cylindrical and Spherical Coordinates

As mentioned in the previous chapter, in order to solve many elasticity problems, formulation must be done in curvilinear coordinates typically using cylindrical or spherical systems. Thus, by following similar methods as used with the strain-displacement relations, we now wish to develop expressions for the equilibrium equations in curvilinear cylindrical and spherical coordinates. By using a direct vector/matrix notation, the equilibrium equations can be expressed as

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{F} = \boldsymbol{0} \tag{3.7.1}$$

where $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$ is the stress matrix or dyadic, $\boldsymbol{e}_i$ are the unit basis vectors in the curvilinear system, and $\boldsymbol{F}$ is the body force vector. The desired curvilinear expressions can be obtained from (3.7.1) by using the appropriate form for $\nabla \cdot \boldsymbol{\sigma}$ from our previous work in Section 1.9.

# Appendix

# C

# 4 Material Behavior—Linear Elastic Solids

The previous two chapters establish elasticity field equations related to the kinematics of small deformation theory and the equilibrium of the associated internal stress field. Based on these physical concepts, three strain-displacement relations (2.2.5), six compatibility equations (2.6.2), and three equilibrium equations (3.6.5) were developed for the general three-dimensional case. Because moment equilibrium simply results in symmetry of the stress tensor, it is not normally included as a separate field equation set. Also, recall that the compatibility equations actually represent only three independent relations, and these equations are needed only to ensure that a given strain field will produce single-valued continuous displacements. Because the displacements are included in the general problem formulation, the solution automatically gives continuous displacements, and the compatibility equations are not formally needed for the general system. Thus, excluding the compatibility relations, it is found that we have now developed nine field equations. The unknowns in these equations include 3 displacement components, 6 components of strain, and 6 stress components, yielding a total of 15 unknowns. Thus, the 9 equations are not sufficient to solve for the 15 unknowns, and additional field equations are needed. This result should not be surprising since up to this point in our development we have not considered the material response. We now wish to complete our general formulation by specializing to a particular material model that provides reasonable characterization of materials under small deformations. The model we will use is that of a linear elastic material, a name that categorizes the entire theory. This chapter presents the basics of the elastic model specializing the formulation for isotropic materials. Related theory for anisotropic media is developed in Chapter 11. Thermoelastic relations are also briefly presented for later use in Chapter 12.

## 4.1 Material Characterization

Relations that characterize the physical properties of materials are called *constitutive equations*. Because of the endless variety of materials and loadings, the study and development of constitutive equations is perhaps one of the most interesting and challenging fields in mechanics. Although continuum mechanics theory has established some principles for systematic development of constitutive equations (Malvern 1969), many constitutive laws have been developed through empirical relations based on experimental evidence. Our interest here is limited to a special class of solid materials with loadings resulting from mechanical or thermal effects. The mechanical behavior of solids is normally defined by constitutive stress-strain relations. Commonly, these relations express the stress as a function of the strain, strain rate, strain history, temperature, and material properties. We choose a rather simple material model called the *elastic solid* that does not include rate or history effects. The model may be described as a deformable continuum that recovers its original configuration when the loadings causing the deformation are removed. Furthermore, we restrict the constitutive stress-strain law to be linear, thus leading to a *linear elastic solid*. Although these assumptions greatly simplify the model, linear elasticity predictions have shown good agreement with experimental data and have provided useful methods to conduct stress analysis. Many structural materials including metals, plastics, ceramics, wood, rock, concrete, and so forth exhibit linear elastic behavior under small deformations.

As mentioned, experimental testing is commonly employed in order to characterize the mechanical behavior of real materials. One such technique is the simple tension test in which a specially prepared cylindrical or flat stock sample is loaded axially in a testing machine. Strain is determined by the change in length between prescribed reference marks on the sample and is usually measured by a clip gage. Load data collected from a load cell is divided by the cross-sectional area in the test section to calculate the stress. Axial stress-strain data is recorded and plotted using standard experimental techniques. Typical qualitative data for three types of structural metals (mild steel, aluminum, cast iron) are shown in Figure 4-1. It is observed that each material exhibits an initial stress-strain response for small deformation that is approximately linear. This is followed by a change to nonlinear behavior that can lead to large deformation, finally ending with sample failure.

For each material the initial linear response ends at a point normally referred to as the *proportional limit*. Another observation in this initial region is that if the loading is removed, the sample returns to its original shape and the strain disappears. This characteristic is the primary descriptor of elastic behavior. However, at some point on the stress-strain curve unloading does not bring the sample back to zero strain and some permanent plastic deformation results. The point at which this nonelastic behavior begins is called the *elastic limit*. Although some materials exhibit different elastic and proportional limits, many times these values are taken to be approximately the same. Another demarcation on the stress-strain curve is referred to as the *yield point,* defined by the location where large plastic deformation begins.

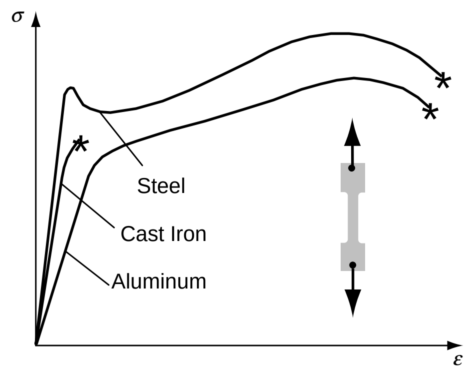



**FIGURE 4-1** *Typical uniaxial stress-strain curves for three structural metals.*

Because mild steel and aluminum are ductile materials, their stress-strain response indicates extensive plastic deformation, and during this period the sample dimensions will be changing. In particular the sample's cross-sectional area undergoes significant reduction, and the stress calculation using division by the original area will now be in error. This accounts for the reduction in the stress at large strain. If we were to calculate the load divided by the true area, the *true stress* would continue to increase until failure. On the other hand, cast iron is known to be a brittle material, and thus its stress-strain response does not show large plastic deformation. For this material, very little nonelastic or nonlinear behavior is observed. It is therefore concluded from this and many other studies that a large variety of real materials exhibits linear elastic behavior under small deformations. This would lead to a linear constitutive model for the one-dimensional axial loading case given by the relation $\sigma = E\varepsilon$, where $E$ is the slope of the uniaxial stress-strain curve. We now use this simple concept to develop the general three-dimensional forms of the linear elastic constitutive model.

## 4.2 Linear Elastic Materials—Hooke's Law

Based on observations from the previous section, in order to construct a general three-dimensional constitutive law for linear elastic materials, we assume that each stress component is linearly related to each strain component

$$
\begin{aligned}
\sigma_x &= C_{11}e_x + C_{12}e_y + C_{13}e_z + 2C_{14}e_{xy} + 2C_{15}e_{yz} + 2C_{16}e_{zx} \\
\sigma_y &= C_{21}e_x + C_{22}e_y + C_{23}e_z + 2C_{24}e_{xy} + 2C_{25}e_{yz} + 2C_{26}e_{zx} \\
\sigma_z &= C_{31}e_x + C_{32}e_y + C_{33}e_z + 2C_{34}e_{xy} + 2C_{35}e_{yz} + 2C_{36}e_{zx} \\
\tau_{xy} &= C_{41}e_x + C_{42}e_y + C_{43}e_z + 2C_{44}e_{xy} + 2C_{45}e_{yz} + 2C_{46}e_{zx} \\
\tau_{yz} &= C_{51}e_x + C_{52}e_y + C_{53}e_z + 2C_{54}e_{xy} + 2C_{55}e_{yz} + 2C_{56}e_{zx} \\
\tau_{zx} &= C_{61}e_x + C_{62}e_y + C_{63}e_z + 2C_{64}e_{xy} + 2C_{65}e_{yz} + 2C_{66}e_{zx}
\end{aligned}
\tag{4.2.1}
$$

where the coefficients $C_{ij}$ are material parameters and the factors of 2 arise because of the symmetry of the strain. Note that this relation could also be expressed by writing the strains as a linear function of the stress components. These relations can be cast into a matrix format as

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{16} \\
C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
C_{61} & \cdot & \cdot & \cdot & \cdot & C_{66}
\end{bmatrix}
\begin{bmatrix} e_x \\ e_y \\ e_z \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix}
\tag{4.2.2}
$$

Relations (4.2.1) can also be expressed in standard tensor notation by writing

$$
\sigma_{ij} = C_{ijkl}e_{kl} \tag{4.2.3}
$$

where $C_{ijkl}$ is a *fourth-order elasticity tensor* whose components include all the material parameters necessary to characterize the material. Based on the symmetry of the stress and strain tensors, the elasticity tensor must have the following properties (see Exercise 4-1):

$$C_{ijkl} = C_{jikl}$$
$$C_{ijkl} = C_{ijlk} \tag{4.2.4}$$

In general, the fourth-order tensor $C_{ijkl}$ has 81 components. However, relations (4.2.4) reduce the number of independent components to 36, and this provides the required match with form (4.2.1) or (4.2.2). Later in Chapter 6 we introduce the concept of strain energy, and this leads to a further reduction to 21 independent elastic components. The components of $C_{ijkl}$ or equivalently $C_{ij}$ are called *elastic moduli* and have units of stress (force/area). In order to continue further, we must address the issues of material homogeneity and isotropy.

If the material is homogenous, the elastic behavior does not vary spatially, and thus all elastic moduli are constant. For this case, the elasticity formulation is straightforward, leading to the development of many analytical solutions to problems of engineering interest. A homogenous assumption is an appropriate model for most structural applications, and thus we primarily choose this particular case for subsequent formulation and problem solution. However, there are a couple of important nonhomogeneous applications that warrant further discussion.

Studies in geomechanics have found that the material behavior of soil and rock commonly depends on distance below the earth's surface. In order to simulate particular geomechanics problems, researchers have used nonhomogeneous elastic models applied to semi-infinite domains. Typical applications have involved modeling the response of a semi-infinite soil mass under surface or subsurface loadings with variation in elastic moduli with depth (see the review by Poulos and Davis 1974). Another more recent application involves the behavior of *functionally graded materials* (FGM) (see Erdogan 1995 and Parameswaran and Shukla 1999, 2002). FGMs are a new class of engineered materials developed with spatially varying properties to suit particular applications. The graded composition of such materials is commonly established and controlled using powder metallurgy, chemical vapor deposition, or centrifugal casting. Typical analytical studies of these materials have assumed linear, exponential, and power-law variation in elastic moduli of the form

$$\begin{aligned} C_{ij}(x) &= C^o_{ij}(1 + ax) \\ C_{ij}(x) &= C^o_{ij}e^{ax} \\ C_{ij}(x) &= C^o_{ij}x^a \end{aligned} \tag{4.2.5}$$

where $C^o_{ij}$ and $a$ are prescribed constants and $x$ is the spatial coordinate. Further investigation of formulation results for such spatially varying moduli are included in Exercises 5-6 and 7-12 in subsequent chapters.

Similar to homogeneity, another fundamental material property is *isotropy*. This property has to do with differences in material moduli with respect to orientation. For example, many materials including crystalline minerals, wood, and fiber-reinforced composites have different elastic moduli in different directions. Materials such as these are said to be *anisotropic*. Note that for most real anisotropic materials there exist particular directions where the properties are the same. These directions indicate *material symmetries*. However, for many engineering materials (most structural metals and many plastics), the orientation of crystalline and grain microstructure is distributed randomly so that macroscopic elastic properties are found to be essentially the same in all directions. Such materials with complete symmetry are called *isotropic*. As expected, an anisotropic model complicates the formulation and solution of problems. We therefore postpone development of such solutions until Chapter 11 and continue our current development under the assumption of isotropic material behavior.

The tensorial form (4.2.3) provides a convenient way to establish the desired isotropic stress-strain relations. If we assume isotropic behavior, the elasticity tensor must be the same under all rotations of the coordinate system. Using the basic transformation properties from relation $(1.5.1)_5$, the fourth-order elasticity tensor must satisfy

$$C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq}$$

It can be shown (Chandrasekharaiah and Debnath 1994) that the most general form that satisfies this isotropy condition is given by

$$C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \tag{4.2.6}$$

where $\alpha, \beta$, and $\gamma$ are arbitrary constants. Verification of the isotropy property of form (4.2.6) is left as Exercise 1-9. Using the general form (4.2.6) in stress-strain relation (4.2.3) gives

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{4.2.7}$$

where we have relabeled particular constants using $\lambda$ and $\mu$. The elastic constant $\lambda$ is called *Lamé's constant,* and $\mu$ is referred to as the *shear modulus* or *modulus of rigidity*. Some texts use the notation $G$ for the shear modulus. Equation (4.2.7) can be written out in individual scalar equations as

$$\begin{aligned}
\sigma_x &= \lambda(e_x + e_y + e_z) + 2\mu e_x \\
\sigma_y &= \lambda(e_x + e_y + e_z) + 2\mu e_y \\
\sigma_z &= \lambda(e_x + e_y + e_z) + 2\mu e_z \\
\tau_{xy} &= 2\mu e_{xy} \\
\tau_{yz} &= 2\mu e_{yz} \\
\tau_{zx} &= 2\mu e_{zx}
\end{aligned} \tag{4.2.8}$$

Relations (4.2.7) or (4.2.8) are called the *generalized Hooke's law for linear isotropic elastic solids*. They are named after Robert Hooke who in 1678 first proposed that the deformation of an elastic structure is proportional to the applied force. Notice the significant simplicity of the isotropic form when compared to the general stress-strain law originally given by (4.2.1). It should be noted that only *two independent elastic constants* are needed to describe the behavior of isotropic materials. As shown in Chapter 11, additional numbers of elastic moduli are needed in the corresponding relations for anisotropic materials.

Stress-strain relations (4.2.7) or (4.2.8) may be inverted to express the strain in terms of the stress. In order to do this it is convenient to use the index notation form (4.2.7) and set the two free indices the same (contraction process) to get

$$\sigma_{kk} = (3\lambda + 2\mu)e_{kk} \tag{4.2.9}$$

This relation can be solved for $e_{kk}$ and substituted back into (4.2.7) to get

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right)$$

which is more commonly written as

$$e_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij} \tag{4.2.10}$$

where $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ and is called *the modulus of elasticity* or *Young's modulus*, and $v = \lambda/[2(\lambda + \mu)]$ is referred to as *Poisson's ratio*. The index notation relation (4.2.10) may be written out in component (scalar) form giving the six equations

$$\begin{aligned}
e_x &= \frac{1}{E}\left[\sigma_x - v(\sigma_y + \sigma_z)\right] \\
e_y &= \frac{1}{E}\left[\sigma_y - v(\sigma_z + \sigma_x)\right] \\
e_z &= \frac{1}{E}\left[\sigma_z - v(\sigma_x + \sigma_y)\right] \\
e_{xy} &= \frac{1+v}{E}\tau_{xy} = \frac{1}{2\mu}\tau_{xy} \\
e_{yz} &= \frac{1+v}{E}\tau_{yz} = \frac{1}{2\mu}\tau_{yz} \\
e_{zx} &= \frac{1+v}{E}\tau_{zx} = \frac{1}{2\mu}\tau_{zx}
\end{aligned} \tag{4.2.11}$$

Constitutive form (4.2.10) or (4.2.11) again illustrates that only two elastic constants are needed to formulate Hooke's law for isotropic materials. By using any of the isotropic forms of Hooke's law, it can be shown that the principal axes of stress coincide with the principal axes of strain (see Exercise 4-4). This result also holds for some but not all anisotropic materials.

# 4.3 Physical Meaning of Elastic Moduli

For the isotropic case, the previously defined elastic moduli have simple physical meaning. These can be determined through investigation of particular states of stress commonly used in laboratory materials testing as shown in Figure 4-2.

## 4.3.1 Simple Tension

Consider the simple tension test as discussed previously with a sample subjected to tension in the $x$ direction (see Figure 4-2). The state of stress is closely represented by the one-dimensional field

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using this in relations (4.2.10) gives a corresponding strain field

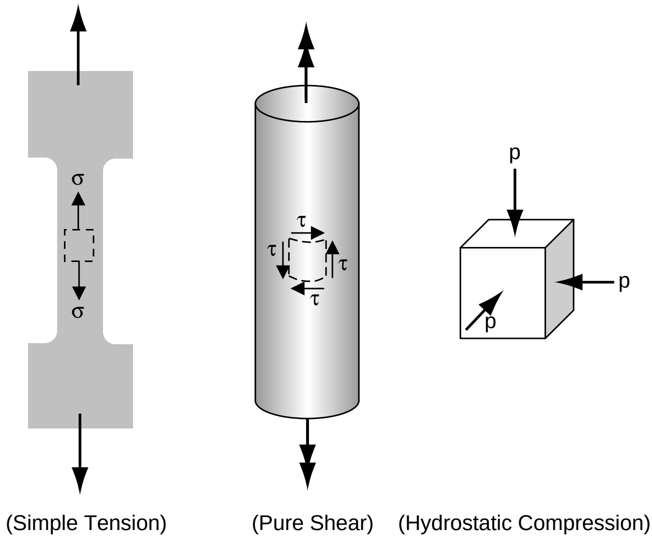



**FIGURE 4-2** *Special characterization states of stress.*

$$e_{ij} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\frac{v}{E}\sigma & 0 \\ 0 & 0 & -\frac{v}{E}\sigma \end{bmatrix}$$

Therefore, $E = \sigma/e_x$ and is simply the slope of the stress-strain curve, while $v = -e_y/e_x = -e_z/e_x$ is the ratio of the transverse strain to the axial strain. Standard measurement systems can easily collect axial stress and transverse and axial strain data, and thus through this one type of test both elastic constants can be determined for materials of interest.

### 4.3.2 Pure Shear

If a thin-walled cylinder is subjected to torsional loading (as shown in Figure 4-2), the state of stress on the surface of the cylindrical sample is given by

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, by using Hooke's law, the corresponding strain field becomes

$$e_{ij} = \begin{bmatrix} 0 & \tau/2\mu & 0 \\ \tau/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus the shear modulus is given by $\mu = \tau/2e_{xy} = \tau/\gamma_{xy}$, and this modulus is simply the slope of the shear stress-shear strain curve.

### 4.3.3 Hydrostatic Compression (or Tension)

The final example is associated with the uniform compression (or tension) loading of a cubical specimen, as shown in Figure 4-2. This type of test would be realizable if the sample was placed in a high-pressure compression chamber. The state of stress for this case is given by

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\delta_{ij}$$

This is an isotropic state of stress and the strains follow from Hooke's law

$$e_{ij} = \begin{bmatrix} -\frac{1-2\nu}{E}p & 0 & 0 \\ 0 & -\frac{1-2\nu}{E}p & 0 \\ 0 & 0 & -\frac{1-2\nu}{E}p \end{bmatrix}$$

The dilatation that represents the change in material volume (see Exercise 2-11) is thus given by $\vartheta = e_{kk} = -3(1-2\nu)p/E$, which can be written as

$$p = -k\vartheta \tag{4.3.1}$$

where $k = E/[3(1-2\nu)]$ is called the *bulk modulus of elasticity*. This additional elastic constant represents the ratio of pressure to the dilatation, which could be referred to as the volumetric stiffness of the material. Notice that as Poisson's ratio approaches 0.5, the bulk modulus becomes unbounded and the material does not undergo any volumetric deformation and hence is incompressible.

Our discussion of elastic moduli for isotropic materials has led to the definition of five constants $\lambda$, $\mu$, $E$, $\nu$, and $k$. However, keep in mind that only two of these are needed to characterize the material. Although we have developed a few relationships between various moduli, many other such relations can also be found. In fact, it can be shown that all five elastic constants are interrelated, and if any two are given, the remaining three can be determined by using simple formulae. Results of these relations are conveniently summarized in Table 4-1. This table should be marked for future reference, because it will prove to be useful for calculations throughout the text.

Typical nominal values of elastic constants for particular engineering materials are given in Table 4-2. These moduli represent average values, and some variation will occur for specific materials. Further information and restrictions on elastic moduli require strain energy concepts, which are developed in Chapter 6.

Before concluding this section, we wish to discuss the forms of Hooke's law in curvilinear coordinates. Previous chapters have mentioned that cylindrical and spherical coordinates (see Figures 1-4 and 1-5) are used in many applications for problem solution. Figures 3-9 and 3-10 defined the stress components in each curvilinear system. In regards to these figures, it follows that the orthogonal curvilinear coordinate directions can be obtained from a base Cartesian system through a simple rotation of the coordinate frame. For isotropic materials, the elasticity tensor $C_{ijkl}$ is the same in all coordinate frames, and thus *the structure of Hooke's law remains the same in any orthogonal curvilinear system*. Therefore, form (4.2.8) can be expressed in cylindrical and spherical coordinates as

**TABLE 4-1** Relations Among Elastic Constants

| | $E$ | $\nu$ | $k$ | $\mu$ | $\lambda$ |
|---|---|---|---|---|---|
| $E,\nu$ | $E$ | $\nu$ | $\dfrac{E}{3(1-2\nu)}$ | $\dfrac{E}{2(1+\nu)}$ | $\dfrac{E\nu}{(1+\nu)(1-2\nu)}$ |
| $E,k$ | $E$ | $\dfrac{3k-E}{6k}$ | $k$ | $\dfrac{3kE}{9k-E}$ | $\dfrac{3k(3k-E)}{9k-E}$ |
| $E,\mu$ | $E$ | $\dfrac{E-2\mu}{2\mu}$ | $\dfrac{\mu E}{3(3\mu-E)}$ | $\mu$ | $\dfrac{\mu(E-2\mu)}{3\mu-E}$ |
| $E,\lambda$ | $E$ | $\dfrac{2\lambda}{E+\lambda+R}$ | $\dfrac{E+3\lambda+R}{6}$ | $\dfrac{E-3\lambda+R}{4}$ | $\lambda$ |
| $\nu,k$ | $3k(1-2\nu)$ | $\nu$ | $k$ | $\dfrac{3k(1-2\nu)}{2(1+\nu)}$ | $\dfrac{3k\nu}{1+\nu}$ |
| $\nu,\mu$ | $2\mu(1+\nu)$ | $\nu$ | $\dfrac{2\mu(1+\nu)}{3(1-2\nu)}$ | $\mu$ | $\dfrac{2\mu\nu}{1-2\nu}$ |
| $\nu,\lambda$ | $\dfrac{\lambda(1+\nu)(1-2\nu)}{\nu}$ | $\nu$ | $\dfrac{\lambda(1+\nu)}{3\nu}$ | $\dfrac{\lambda(1-2\nu)}{2\nu}$ | $\lambda$ |
| $k,\mu$ | $\dfrac{9k\mu}{6k+\mu}$ | $\dfrac{3k-2\mu}{6k+2\mu}$ | $k$ | $\mu$ | $k-\dfrac{2}{3}\mu$ |
| $k,\lambda$ | $\dfrac{9k(k-\lambda)}{3k-\lambda}$ | $\dfrac{\lambda}{3k-\lambda}$ | $k$ | $\dfrac{3}{2}(k-\lambda)$ | $\lambda$ |
| $\mu,\lambda$ | $\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ | $\dfrac{\lambda}{2(\lambda+\mu)}$ | $\dfrac{3\lambda+2\mu}{3}$ | $\mu$ | $\lambda$ |

$R=\sqrt{E^2+9\lambda^2+2E\lambda}$

$$\begin{aligned}
\sigma_r &= \lambda(e_r+e_\theta+e_z)+2\mu e_r & \sigma_R &= \lambda(e_R+e_\phi+e_\theta)+2\mu e_R \\
\sigma_\theta &= \lambda(e_r+e_\theta+e_z)+2\mu e_\theta & \sigma_\phi &= \lambda(e_R+e_\phi+e_\theta)+2\mu e_\phi \\
\sigma_z &= \lambda(e_r+e_\theta+e_z)+2\mu e_z & \sigma_\theta &= \lambda(e_R+e_\phi+e_\theta)+2\mu e_\theta \\
\tau_{r\theta} &= 2\mu e_{r\theta} & \tau_{R\phi} &= 2\mu e_{R\phi} \\
\tau_{\theta z} &= 2\mu e_{\theta z} & \tau_{\phi\theta} &= 2\mu e_{\phi\theta} \\
\tau_{zr} &= 2\mu e_{zr} & \tau_{\theta R} &= 2\mu e_{\theta R}
\end{aligned} \tag{4.3.2}$$

The complete set of elasticity field equations in each of these coordinate systems is given in Appendix A.

## 4.4 Thermoelastic Constitutive Relations

It is well known that a temperature change in an unrestrained elastic solid produces deformation. Thus, a general strain field results from both mechanical and thermal effects. Within the context of linear small deformation theory, the total strain can be decomposed into the sum of mechanical and thermal components as

# Appendix

# D

Show that the invariants of $P$ are $I_P = II_P = 2$, $III_P = 0$, and find the eigenvalues and eigenvectors of $P$.

2. Using a procedure similar to that employed in Equations (2.41–42), obtain transformation equations for the components of third- and fourth-order tensors in two sets of bases $e_i$ and $e'_i$ that are related by the 3-D transformation tensor $Q$ with components $Q_{ij} = e_i \cdot e'_j$.
3. Given any second-order tensor $S$ linearize the expression $S^2 = SS$ in the direction of an increment $U$.
4. Consider a functional $I$ that when applied to the function $y(x)$ gives the integral:

$$I(y(x)) = \int_a^b f(x, y, y')\,dx$$

where $f$ is a general expression involving $x$, $y(x)$ and the derivative $y'(x) = dy/dx$. Show that the function $y(x)$ that renders the above functional stationary and satisfies the boundary conditions $y(a) = y_a$ and $y(b) = y_b$ is the solution of the following Euler–Lagrange differential equation:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

5. Prove Equations (2.135a–g) following the procedure shown in Example 2.10.
6. Show that the volume of a closed 3-D body $V$ is variously given as,

$$V = \int_{\partial V} n_x\,dA = \int_{\partial V} n_y\,dA = \int_{\partial V} n_z\,dA$$

where $n_x$, $n_y$ and $n_z$ are the $x$, $y$ and $z$ components of the unit normal $n$.

CHAPTER THREE

# KINEMATICS

## 3.1 INTRODUCTION

It is almost a tautology to say that a proper description of motion is fundamental to finite deformation analysis, but such an emphasis is necessary because infinitesimal deformation analysis implies a host of assumptions that we take for granted and seldom articulate. For example, we have seen in Chapter 1, in the simple truss example, that care needs to be exercised when large deformations are anticipated and that a linear definition of strain is totally inadequate in the context of a finite rotation. A study of finite deformation will require that cherished assumptions be abandoned and a fresh start made with an open (but not empty!) mind.

Kinematics is the study of motion and deformation without reference to the cause. We shall see immediately that consideration of finite deformation enables alternative coordinate systems to be employed, namely, material and spatial descriptions associated with the names of Lagrange and Euler respectively.

Although we are not directly concerned with inertial effects, nevertheless time derivatives of various kinematic quantities enrich our understanding and also provide the basis for the formulation of the virtual work expression of equilibrium, which uses the notion of virtual velocity and associated kinematic quantities.

Wherever appropriate, nonlinear kinematic quantities are linearized in preparation for inclusion in the linearized equilibrium equations that form the basis of the Newton–Raphson solution to the finite element equilibrium equations.

## 3.2 THE MOTION

Figure 3.1 shows the general motion of a deformable body. The body is imagined as being an assemblage of material particles that are labeled by the coordinates $X$,

| Time | Length | Area | Volume | Density |
|---|---|---|---|---|
| 0 | $S$ | $A$ | $V$ | $\rho_0$ |
| $t$ | $s$ | $a$ | $v$ | $\rho$ |

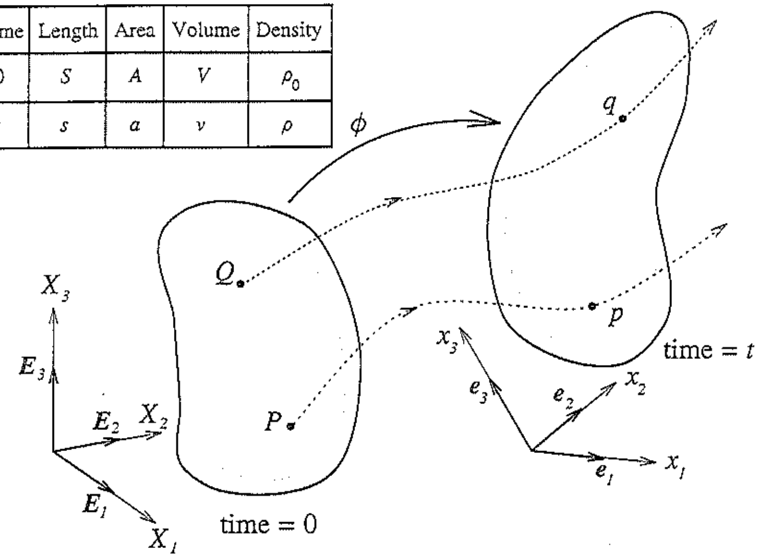



**FIGURE 3.1** General motion of a deformable body.

with respect to Cartesian basis $\boldsymbol{E}_I$, at their initial positions at time $t = 0$. Generally the current positions of these particles are located, at time $= t$, by the coordinates $\boldsymbol{x}$ with respect to an alternative Cartesian basis $\boldsymbol{e}_i$. In the remainder of this text the bases $\boldsymbol{E}_I$ and $\boldsymbol{e}_i$ will be taken to be coincident. However the notational distinction between $\boldsymbol{E}_I$ and $\boldsymbol{e}_i$ will be retained in order to identify the association of quantities with initial or current configurations. The motion can be mathematically described by a mapping $\phi$ between initial and current particle positions as,

$$\boldsymbol{x} = \phi(\boldsymbol{X}, t) \tag{3.1}$$

For a fixed value of $t$ the above equations represent a mapping between the undeformed and deformed bodies. Additionally, for a fixed particle $\boldsymbol{X}$, Equation (3.1) describes the motion or trajectory of this particle as a function of time. In finite deformation analysis no assumptions are made regarding the magnitude of the displacement $\boldsymbol{x} - \boldsymbol{X}$, indeed the displacement may well be of the order or even exceed the initial dimensions of the body as is the case, for example, in metal forming. In infinitesimal deformation analysis the displacement $\boldsymbol{x} - \boldsymbol{X}$ is assumed to be small in comparison with the dimensions of the body, and geometrical changes are ignored.



## 3.3 MATERIAL AND SPATIAL DESCRIPTIONS

In finite deformation analysis a careful distinction has to be made between the coordinate systems that can be chosen to describe the behavior of the body whose motion is under consideration. Roughly speaking, relevant quantities, such as density, can be described in terms of where the body was before deformation or where it is during deformation; the former is called a *material* description, and the latter is called a *spatial* description. Alternatively these are often referred to as *Lagrangian* and *Eulerian* descriptions respectively. A material description refers to the behavior of a material particle, whereas a spatial description refers to the behaviour at a spatial position. Nevertheless irrespective of the description eventually employed, the governing equations must obviously refer to where the body is and hence must primarily be formulated using a spatial description.

Fluid mechanicians almost exclusively work in terms of a spatial description because it is not appropriate to describe the behavior of a material particle in, for example, a steady-state flow situation. Solid mechanicians, on the other hand, will generally at some stage of a formulation have to consider the constitutive behavior of the material particle, which will involve a material description. In many instances – for example, polymer flow – where the behavior of the flowing material may be time-dependent, these distinctions are less obvious.

In order to understand the difference between a material and spatial description, consider a simple scalar quantity such as the material density $\rho$:

(a) *Material description*: the variation of $\rho$ over the body is described with respect to the original (or initial) coordinate $\boldsymbol{X}$ used to label a material particle in the continuum at time $t = 0$ as,

$$\rho = \rho(\boldsymbol{X}, t) \tag{3.2a}$$

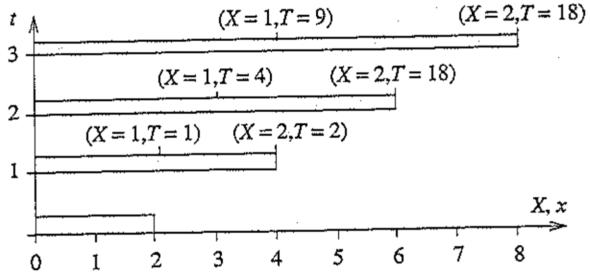
(b) *Spatial description*: $\rho$ is described with respect to the position in space, $\boldsymbol{x}$, currently occupied by a material particle in the continuum at time $t$ as,

$$\rho = \rho(\boldsymbol{x}, t) \tag{3.2b}$$

In Equation (3.2a) a change in time $t$ implies that the same material particle $\boldsymbol{X}$ has a different density $\rho$. Consequently interest is focused on the material particle $\boldsymbol{X}$. In Equation (3.2b), however, a change in the time $t$ implies that a different density is observed at the same spatial position $\boldsymbol{x}$, now probably occupied by a different particle. Consequently interest is focused on a spatial position $\boldsymbol{x}$.

**EXAMPLE 3.1: Uniaxial motion**

This example illustrates the difference between a material and a spatial description of motion. Consider the mapping $x = (1+t)X$ defining the motion of a rod of initial length two units. The rod experiences a temperature distribution given by the material description $T = Xt^2$ or by the spatial description $T = xt^2/(1+t)$, see diagram below.




The diagram makes it clear that the particle material coordinates (label) $X$ remains associated with the particle while its spatial position $x$ changes. The temperature at a given time can be found in two ways, for example, at time $t = 3$ the temperature of the particle labeled $X = 2$ is $T = 2 \times 3^2 = 18$. Alternatively the temperature of the same particle which at $t = 3$ is at the spatial position $x = 8$ is $T = 8 \times 3^2/(1+3) = 18$. Note that whatever the time it makes no sense to enquire about particles for which $X > 2$, nor, for example, at time $t = 3$ does it make sense to enquire about the temperature at $x > 8$.

Often it is necessary to transform between the material and spatial descriptions for relevant quantities. For instance, given a scalar quantity, such as the density, a material description can be easily obtained from a spatial description by using motion Equation (3.1) as,

$$\rho(X, t) = \rho(\phi(X, t), t) \tag{3.2c}$$

Certain magnitudes, irrespective of whether they are materially or spatially described, are naturally associated with the current or initial configurations of the body. For instance the initial density of the body is a material magnitude, whereas the current density is intrinsically a spatial quantity. Nevertheless, Equations (3.2a–c)



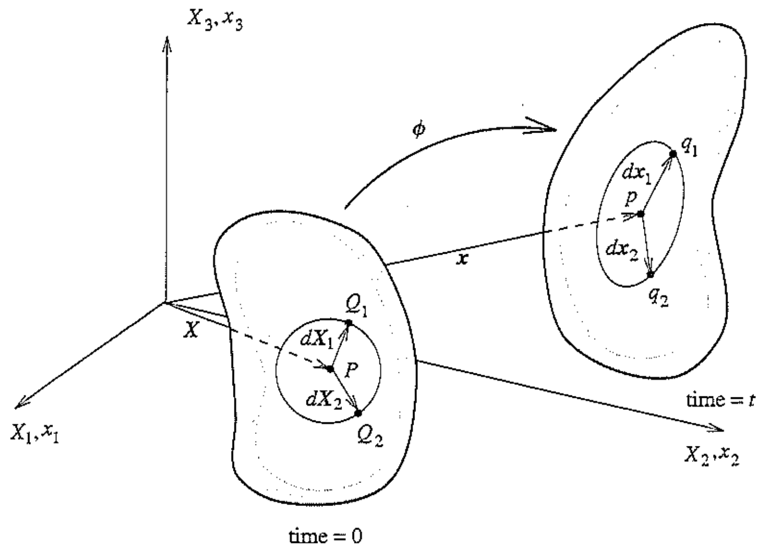



**FIGURE 3.2** General motion in the neighborhood of a particle.

clearly show that spatial quantities can, if desired, be expressed in terms of the initial coordinates.

## 3.4 DEFORMATION GRADIENT

A key quantity in finite deformation analysis is the deformation gradient $F$, which is involved in all equations relating quantities before deformation to corresponding quantities after (or during) deformation. The deformation gradient tensor enables the relative spatial position of two neighboring particles after deformation to be described in terms of their relative material position before deformation; consequently, it is central to the description of deformation and hence strain.

Consider two material particles $Q_1$ and $Q_2$ in the neighborhood of a material particle $P$: see Figure 3.2. The positions of $Q_1$ and $Q_2$ relative to $P$ are given by the elemental vectors $dX_1$ and $dX_2$ as,

$$dX_1 = X_{Q_1} - X_P; \qquad dX_2 = X_{Q_2} - X_P \tag{3.3a,b}$$

After deformation the material particles $P$, $Q_1$, and $Q_2$ have deformed to current spatial positions given by the mapping (3.1) as,

$$x_p = \phi(X_P, t); \qquad x_{q_1} = \phi(X_{Q_1}, t); \qquad x_{q_2} = \phi(X_{Q_2}, t) \tag{3.4a,b,c}$$

and the corresponding elemental vectors become,

$$dx_1 = x_{q_1} - x_p = \phi(X_P + dX_1, t) - \phi(X_P, t) \tag{3.5a}$$

$$dx_2 = x_{q_2} - x_p = \phi(X_P + dX_2, t) - \phi(X_P, t) \tag{3.5b}$$

Defining the *deformation gradient tensor* $F$ as,

$$F = \frac{\partial \phi}{\partial X} = \nabla_0 \phi \tag{3.6}$$

then the elemental vectors $dx_1$ and $dx_2$ can be obtained in terms of $dX_1$ and $dX_2$ as,

$$dx_1 = F dX_1; \qquad dx_2 = F dX_2 \tag{3.7a,b}$$

Note that $F$ transforms vectors in the initial or reference configuration into vectors in the current configuration and is therefore said to be a *two-point* tensor.

*Remark 1:* In many textbooks the motion is expressed as,

$$x = x(X, t) \tag{3.8}$$

which allows the deformation gradient tensor to be written, perhaps, in a clearer manner as,

$$F = \frac{\partial x}{\partial X} \tag{3.9a}$$

In indicial notation the deformation gradient tensor is expressed as,

$$F = \sum_{i,I=1}^{3} F_{iI} e_i \otimes E_I; \qquad F_{iI} = \frac{\partial x_i}{\partial X_I}; \qquad i, I = 1, 2, 3 \tag{3.9b}$$

where lowercase indices refer to current (spatial) Cartesian coordinates, whereas uppercase indices refer to initial (material) Cartesian coordinates.

Confining attention to a single elemental material vector $dX$, the corresponding vector $dx$ in the spatial configuration is conveniently written as,

$$dx = F dX \tag{3.10}$$

The inverse of $F$ is,

$$F^{-1} = \frac{\partial X}{\partial x} = \nabla \phi^{-1} \tag{3.11a}$$

which in indicial notation is,

$$F^{-1} = \sum_{I,i=1}^{3} \frac{\partial X_I}{\partial x_i} E_I \otimes e_i \tag{3.11b}$$



*Remark 2:* Much research literature expresses the relationship between quantities in the material and spatial configurations in terms of the general concepts of *push forward* and *pull back*. For example, the elemental spatial vector $dx$ can be considered as the push forward equivalent of the material vector $dX$. This can be expressed in terms of the operation,

$$dx = \phi_*[dX] = F dX \tag{3.12}$$

Inversely, the material vector $dX$ is the pull back equivalent of the spatial vector $dx$, which is expressed as*,

$$dX = \phi_*^{-1}[dx] = F^{-1} dx \tag{3.13}$$

Observe that in (3.12) the nomenclature $\phi_*[\ ]$ implies an *operation* that will be evaluated in different ways for different operands [ ].

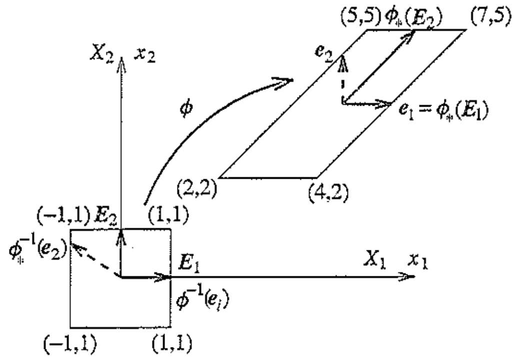
### EXAMPLE 3.2: Uniform deformation

This example illustrates the role of the deformation gradient tensor $F$. Consider the uniform deformation given by the mapping,

$$x_1 = \frac{1}{4}(18 + 4X_1 + 6X_2)$$

$$x_2 = \frac{1}{4}(14 + 6X_2)$$

which, for a square of side two units initially centred at $X = (0, 0)$, produces the deformation show below.




*(continued)*

* In the literature $\phi_*[\ ]$ and $\phi_*^{-1}[\ ]$ are often written, as $\phi_*$ and $\phi^*$ respectively.

**EXAMPLE 3.2** *(cont.)*

$$F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}; \qquad F^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 \\ 0 & 2 \end{bmatrix}$$

Unit vectors $E_1$ and $E_2$ in the initial configuration deform to,

$$\phi_*[E_1] = F\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad \phi_*[E_2] = F\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$

and unit vectors in the current (deformed) configuration deform from,

$$\phi_*^{-1}[e_1] = F^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad \phi_*^{-1}[e_2] = F^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix}$$

## 3.5 STRAIN

As a general measure of deformation, consider the change in the scalar product of the two elemental vectors $dX_1$ and $dX_2$ shown in Figure 3.2 as they deform to $dx_1$ and $dx_2$. This change will involve both the stretching (that is, change in length) and changes in the enclosed angle between the two vectors. Recalling (3.7), the spatial scalar product $dx_1 \cdot dx_2$ can be found in terms of the material vectors $dX_1$ and $dX_2$ as,

$$dx_1 \cdot dx_2 = dX_1 \cdot C\, dX_2 \tag{3.14}$$

where $C$ is the *right Cauchy–Green deformation tensor*, which is given in terms of the deformation gradient as $F$ as,

$$C = F^T F \tag{3.15}$$

Note that in (3.15) the tensor $C$ operates on the material vectors $dX_1$ and $dX_2$ and consequently $C$ is called a material tensor quantity.

Alternatively the initial material scalar product $dX_1 \cdot dX_2$ can be obtained in terms of the spatial vectors $dx_1$ and $dx_2$ via the *left Cauchy–Green* or *Finger tensor* $b$ as,*

$$dX_1 \cdot dX_2 = dx_1 \cdot b^{-1} dx_2 \tag{3.16}$$

where $b$ is,

$$b = F F^T \tag{3.17}$$

* In $C = F^T F$, $F$ is on the right and in $b = FF^T$, $F$ is on the left.



Observe that in (3.16) $b^{-1}$ operates on the spatial vectors $dx_1$ and $dx_2$, and consequently $b^{-1}$, or indeed $b$ itself, is a spatial tensor quantity.

The change in scalar product can now be found in terms of the material vectors $dX_1$ and $dX_2$ and the *Lagrangian* or *Green strain tensor* $E$ as,

$$\frac{1}{2}(dx_1 \cdot dx_2 - dX_1 \cdot dX_2) = dX_1 \cdot E\, dX_2 \tag{3.18a}$$

where the material tensor $E$ is,

$$E = \frac{1}{2}(C - I) \tag{3.18b}$$

Alternatively, the same change in scalar product can be expressed with reference to the spatial elemental vectors $dx_1$ and $dx_2$ and the *Eulerian* or *Almansi strain tensor* $e$ as,

$$\frac{1}{2}(dx_1 \cdot dx_2 - dX_1 \cdot dX_2) = dx_1 \cdot e\, dx_2 \tag{3.19a}$$

where the spatial tensor $e$ is,

$$e = \frac{1}{2}(I - b^{-1}) \tag{3.19b}$$

**EXAMPLE 3.3: Green and Almansi strain tensors**

For the deformation given in Example 3.2 the right and left Cauchy–Green deformation tensors are respectively,

$$C = F^T F = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 3 & 9 \end{bmatrix}; \qquad b = FF^T = \frac{1}{4}\begin{bmatrix} 13 & 9 \\ 9 & 9 \end{bmatrix}$$

from which the Green's strain tensor is simply,

$$E = \frac{1}{4}\begin{bmatrix} 0 & 3 \\ 3 & 7 \end{bmatrix}$$

and the Almansi strain tensor is,

$$e = \frac{1}{18}\begin{bmatrix} 0 & 9 \\ 9 & -4 \end{bmatrix}$$

The physical interpretation of these strain measures will be demonstrated in the next example.

*Remark 3:* The general nature of the scalar product as a measure of deformation can be clarified by taking $dX_2$ and $dX_1$ equal to $dX$ and consequently $dx_1 =$

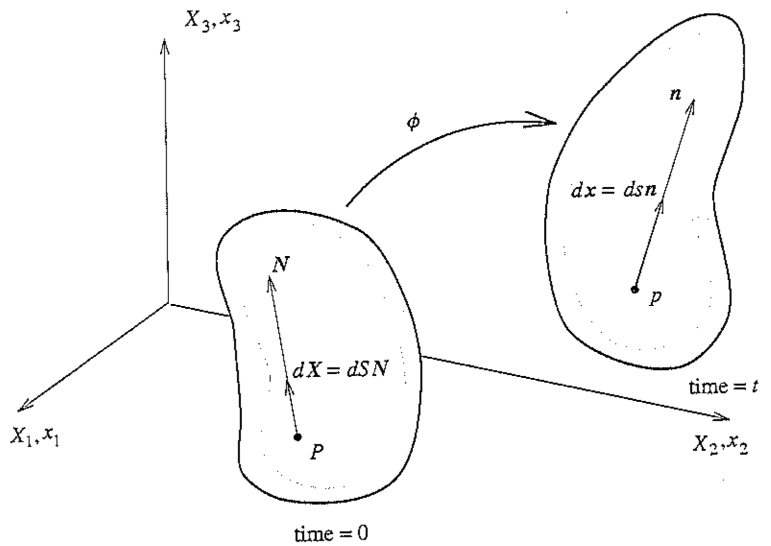



**FIGURE 3.3** Change in length.

$dx_2 = dx$. This enables initial (material) and current (spatial) elemental lengths squared to be determined as (see Figure 3.3),

$$dS^2 = dX \cdot dX; \qquad ds^2 = dx \cdot dx \tag{3.20a,b}$$

The change in the squared lengths that occurs as the body deforms from the initial to the current configuration can now be written in terms of the elemental material vector $dX$ as,

$$\frac{1}{2}(ds^2 - dS^2) = dX \cdot E\, dX \tag{3.21}$$

which, upon division by $dS^2$, gives the scalar Green's strain as,

$$\frac{ds^2 - dS^2}{2\, dS^2} = \frac{dX}{dS} \cdot E \frac{dX}{dS} \tag{3.22}$$

where $dX/dS$ is a unit material vector $N$ in the direction of $dX$, hence, finally,

$$\frac{1}{2}\left(\frac{ds^2 - dS^2}{dS^2}\right) = N \cdot EN \tag{3.23}$$

Using Equation (3.19a) a similar expression involving the Almansi strain tensor



can be derived as,

$$\frac{1}{2}\left(\frac{ds^2 - dS^2}{ds^2}\right) = n \cdot en \tag{3.24}$$

where $n$ is a unit vector in the direction of $dx$.

### EXAMPLE 3.4: Physical interpretation of strain tensors

Refering to Example 3.2 the magnitude of the elemental vector $dx_2$ is $ds_2 = 4.5^{1/2}$. Using (3.23) the scalar value of Green's strain associated with the elemental material vector $dX_2$ is,

$$\varepsilon_G = \frac{1}{2}\left(\frac{ds^2 - dS^2}{dS^2}\right) = \frac{7}{4}$$

Again using (3.23) and Example 3.3 the same strain can be determined from Green's strain tensor $E$ as,

$$\varepsilon_G = N^T EN = [0, 1]\frac{1}{4}\begin{bmatrix} 0 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{7}{4}$$

Using (3.24) the scalar value of the Almansi strain associated with the elemental spatial vector $dx_2$ is,

$$\varepsilon_A = \frac{1}{2}\left(\frac{ds^2 - dS^2}{ds^2}\right) = \frac{7}{18}$$

Alternatively, again using (3.24) and Example 3.3 the same strain is determined from the Almansi strain tensor $e$ as,

$$\varepsilon_A = n^T en = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\frac{1}{18}\begin{bmatrix} 0 & 9 \\ 9 & -4 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{7}{18}$$

*Remark 4:* In terms of the language of pull back and push forward, the material and spatial strain measures can be related through the operator $\phi_*$. Precisely how this operator works in this case can be discovered by recognizing, because of their definitions, the equality,

$$dx_1 \cdot e\, dx_2 = dX_1 \cdot E\, dX_2 \tag{3.25}$$

for any corresponding pairs of elemental vectors. Recalling Equations (3.12–13) enables the push forward and pull back operations to be written as,

Push forward
$$e = \phi_*[E] = F^{-T} E F^{-1} \tag{3.26a}$$

Pull back
$$E = \phi_*^{-1}[e] = F^T e F \tag{3.26b}$$

# Appendix

# E

# CHAPTER FOUR

# STRESS AND EQUILIBRIUM

## 4.1 INTRODUCTION

This chapter will introduce the stress and equilibrium concepts for a deformable body undergoing a finite motion. Stress is first defined in the current configuration in the standard way as force per unit area. This leads to the well-known Cauchy stress tensor as used in linear analysis. We will then derive the differential equations enforcing translational and rotational equilibrium and the equivalent principle of virtual work.

In contrast to linear small displacement analysis, stress quantities that refer back to the initial body configuration can also be defined. This will be achieved using work conjugacy concepts that will lead to the Piola–Kirchhoff stress tensors and alternative equilibrium equations. Finally, the objectivity of several stress rate tensors is considered.

## 4.2 CAUCHY STRESS TENSOR

### 4.2.1 Definition

Consider a general deformable body at its current position as shown in Figure 4.1. In order to develop the concept of stress it is necessary to study the action of the forces applied by one region $R_1$ of the body on the remaining part $R_2$ of the body with which it is in contact. For this purpose consider the element of area $\Delta a$ to normal $\boldsymbol{n}$ in the neighborhood of spatial point $p$ shown in Figure 4.1. If the resultant force on this area is $\Delta \boldsymbol{p}$, the traction vector $\boldsymbol{t}$ corresponding to the normal $\boldsymbol{n}$ at $p$ is





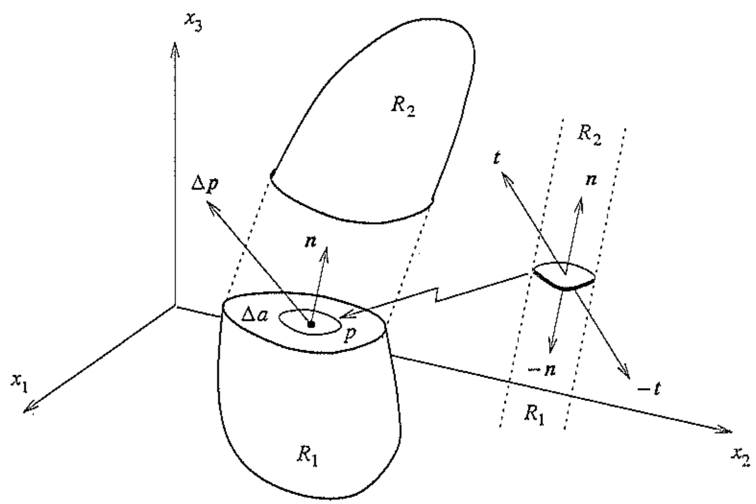



**FIGURE 4.1** Traction vector.

defined as,

$$\boldsymbol{t}(\boldsymbol{n}) = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{p}}{\Delta a} \tag{4.1}$$

where the relationship between $\boldsymbol{t}$ and $\boldsymbol{n}$ must be such that satisfies Newton's third law of action and reaction, which is expressed as (see Figure 4.1),

$$\boldsymbol{t}(-\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{n}) \tag{4.2}$$

To develop the idea of a stress tensor, let the three traction vectors associated with the three Cartesian directions $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_3$ be expressed in a component form as (see Figure 4.2),

$$\boldsymbol{t}(\boldsymbol{e}_1) = \sigma_{11}\boldsymbol{e}_1 + \sigma_{21}\boldsymbol{e}_2 + \sigma_{31}\boldsymbol{e}_3 \tag{4.3a}$$

$$\boldsymbol{t}(\boldsymbol{e}_2) = \sigma_{12}\boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 + \sigma_{32}\boldsymbol{e}_3 \tag{4.3b}$$

$$\boldsymbol{t}(\boldsymbol{e}_3) = \sigma_{13}\boldsymbol{e}_1 + \sigma_{23}\boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3 \tag{4.3c}$$

Although the general equilibrium of a deformable body will be discussed in detail in the next section, a relationship between the traction vector $\boldsymbol{t}$ corresponding to a general direction $\boldsymbol{n}$ and the components $\sigma_{ij}$ can be obtained only by studying the translational equilibrium of the elemental tetrahedron shown in Figure 4.3. Letting $\boldsymbol{f}$ be the force per unit volume acting on the body at point $p$ (which in general

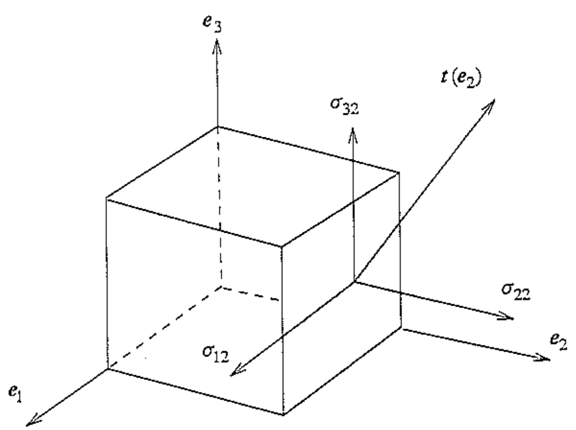



**FIGURE 4.2** Stress components.

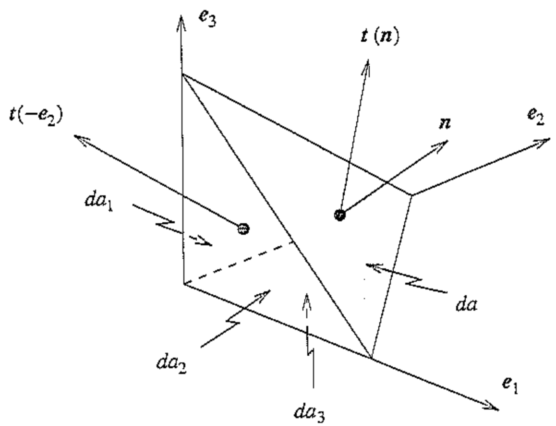



**FIGURE 4.3** Elemental tetrahedron.

could also include inertia terms), the equilibrium of the tetrahedron is given as,

$$t(n)\,da + \sum_{i=1}^{3} t(-e_i)\,da_i + f\,dv = 0 \tag{4.4}$$

where $da_i = (n \cdot e_i)\,da$ is the projection of the area $da$ onto the plane orthogonal to the Cartesian direction $i$ (see Figure 4.3) and $dv$ is the volume of the tetrahedron. Dividing Equation (4.4) by $da$, recalling Newton's third law, using Equations



(4.3a–c), and noting that $dv/da \to 0$ gives,

$$\begin{aligned} t(n) &= -\sum_{j=1}^{3} t(-e_j)\frac{da_j}{da} - f\frac{dv}{da} \\ &= \sum_{j=1}^{3} t(e_j)(n \cdot e_j) \\ &= \sum_{i,j=1}^{3} \sigma_{ij}(e_j \cdot n)\,e_i \end{aligned} \tag{4.5}$$

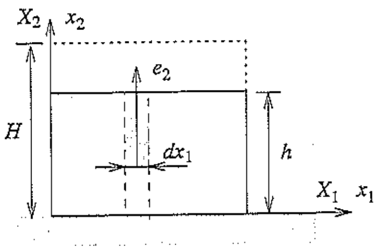
Observing that $(e_j \cdot n)e_i$ can be rewritten in terms of the tensor product as $(e_i \otimes e_j)n$ gives,

$$\begin{aligned} t(n) &= \sum_{i,j=1}^{3} \sigma_{ij}(e_j \cdot n)\,e_i \\ &= \sum_{i,j=1}^{3} \sigma_{ij}(e_i \otimes e_j)n \\ &= \left[\sum_{i,j=1}^{3} \sigma_{ij}(e_i \otimes e_j)\right]n \end{aligned} \tag{4.6}$$

which clearly identifies a tensor $\sigma$, known as the *Cauchy stress tensor*, that relates the normal vector $n$ to the traction vector $t$ as,

$$t(n) = \sigma n; \qquad \sigma = \sum_{i,j=1}^{3} \sigma_{ij}\, e_i \otimes e_j \tag{4.7a,b}$$

**EXAMPLE 4.1: Rectangular block under self-weight (i)**




A simple example of a two-dimensional stress tensor results from the self-weight of a block of uniform initial density $\rho_0$ resting on a frictionless surface as shown in the figure above. For simplicity we will assume that there is no lateral deformation (in linear elasticity this would imply that the Poisson ratio $\nu$=0).

*(continued)*

**EXAMPLE 4.1** *(cont.)*

Using Definition (4.1), the traction vector $t$ associated with the unit vertical vector $e_2$ at an arbitrary point at height $x_2$, initially at height $X_2$, is equal to the weight of material above an infinitesimal section divided by the area of this section. This gives,

$$t(e_2) = \frac{(-\int_y^h \rho g\, dx_2)\, e_2 dx_1}{dx_1}$$

where $g$ is the acceleration of gravity and $h$ is the height of the block after deformation. The mass conservation Equation (3.57) implies that $\rho dx_1 dx_2 = \rho_0 dX_1 dX_2$, which in conjunction with the lack of lateral deformation gives,

$$t(e_2) = \rho_0 g (H - X_2)\, e_2$$

Combining this equation with the fact that the stress components $\sigma_{12}$ and $\sigma_{22}$ are defined in Equation (4.3) by the expression $t(e_2) = \sigma_{12} e_1 + \sigma_{22} e_2$ gives $\sigma_{12} = 0$ and $\sigma_{22} = -\rho_0 g (H - X_2)$. Using a similar process and given the absence of horizontal forces, it is easy to show that the traction vector associated with the horizontal unit vector is zero and consequently $\sigma_{11} = \sigma_{21} = 0$. The complete stress tensor in Cartesian components is therefore,

$$[\sigma] = \begin{bmatrix} 0 & 0 \\ 0 & \rho_0 g (X_2 - H) \end{bmatrix}$$

The Cauchy stress tensor can alternatively be expressed in terms of its principal directions $m_1, m_2, m_3$ and principal stresses $\sigma_{\alpha\alpha}$ for $\alpha = 1, 2, 3$ as,

$$\sigma = \sum_{\alpha=1}^{3} \sigma_{\alpha\alpha}\, m_\alpha \otimes m_\alpha \tag{4.8}$$

where from Equations (2.57a–b), the eigenvectors $m_\alpha$ and eigenvalues $\sigma_{\alpha\alpha}$ satisfy,

$$\sigma m_\alpha = \sigma_{\alpha\alpha} m_\alpha \tag{4.9}$$

In the next chapter we shall show that for isotropic materials the principal directions $m_\alpha$ of the Cauchy stress coincide with the principal Eulerian triad $n_\alpha$ introduced in the previous chapter.

Note that $\sigma$ is a spatial tensor; equivalent material stress measures associated with the initial configuration of the body will be discussed later. Note also that the well-known symmetry of $\sigma$ has not yet been established. In fact this results from the rotational equilibrium equation, which is discussed in the following section.



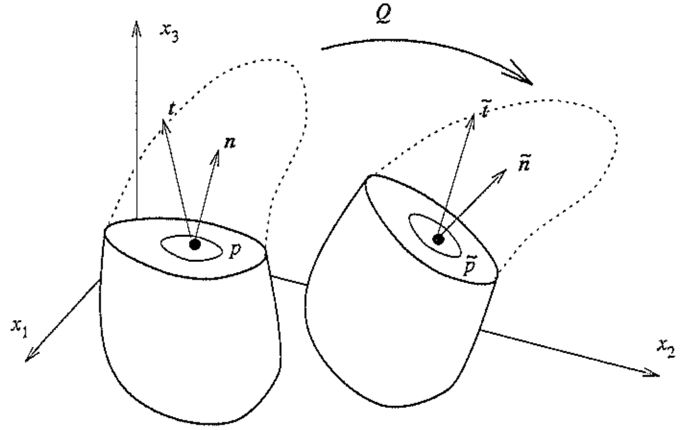



**FIGURE 4.4** Superimposed rigid body motion.

### 4.2.2 Stress Objectivity

Because the Cauchy stress tensor is a key feature of any equilibrium or material equation, it is important to inquire whether $\sigma$ is objective as defined in Section 3.15. For this purpose consider the transformations of the normal and traction vectors implied by the superimposed rigid body motion $Q$ shown in Figure 4.4 as,

$$\tilde{t}(\tilde{n}) = Q t(n) \tag{4.10a}$$

$$\tilde{n} = Q n \tag{4.10b}$$

Using the relationship between the traction vector and stress tensor given by Equation (4.7a) in conjunction with the above equation gives,

$$\tilde{\sigma} = Q \sigma Q^T \tag{4.11}$$

The rotation of $\sigma$ given by the above equation conforms with the definition of objectivity given by Equation (3.135), and hence $\sigma$ is objective and a valid candidate for inclusion in a material description. It will be shown later that the material rate of change of stress is not an objective tensor.

## 4.3 EQUILIBRIUM

### 4.3.1 Translational Equilibrium

In order to derive the differential static equilibrium equations, consider the spatial configuration of a general deformable body defined by a volume $v$ with boundary

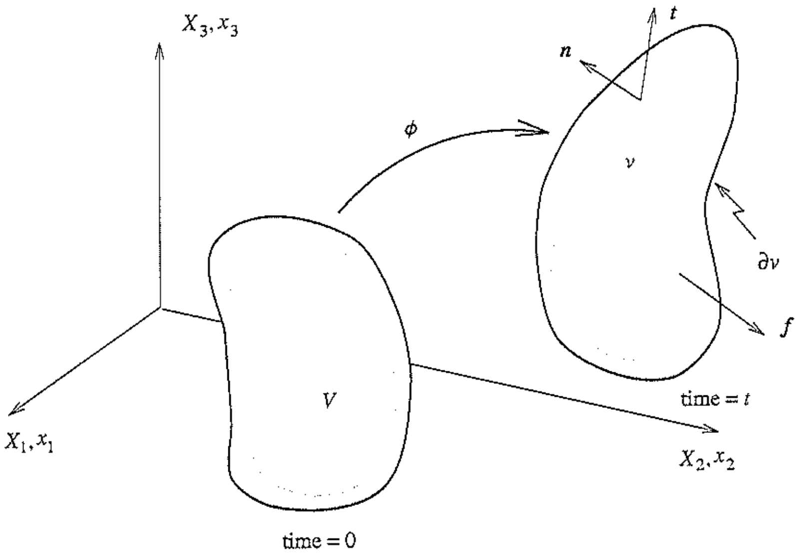



**FIGURE 4.5** Equilibrium.

area $\partial v$ as shown in Figure 4.5. We can assume that the body is under the action of body forces $f$ per unit volume and traction forces $t$ per unit area acting on the boundary. For simplicity, however, inertia forces will be ignored, and therefore translational equilibrium implies that the sum of all forces acting on the body vanishes. This gives,

$$\int_{\partial v} t \, da + \int_v f \, dv = 0 \tag{4.12}$$

Using Equation (4.7a) for the traction vector enables Equation (4.12) to be expressed in terms of the Cauchy stresses as,

$$\int_{\partial v} \sigma n \, da + \int_v f \, dv = 0 \tag{4.13}$$

The first term in this equation can be transformed into a volume integral by using the Gauss theorem given in Equation (2.139) to give,

$$\int_v (\operatorname{div} \sigma + f) \, dv = 0 \tag{4.14}$$

where the vector $\operatorname{div} \sigma$ is defined in Section 2.4.1. The fact that the above equation can be equally applied to any enclosed region of the body implies that the integrand



function must vanish, that is,

$$\operatorname{div} \sigma + f = \mathbf{0} \tag{4.15}$$

This equation is known as the local (that is, pointwise) spatial equilibrium equation for a deformable body. In anticipation of situations during a solution procedure in which equilibrium is not yet satisfied, the above equation defines the pointwise out-of-balance or residual force per unit volume $r$ as,

$$r = \operatorname{div} \sigma + f \tag{4.16}$$

**EXAMPLE 4.2: Rectangular block under self-weight (ii)**

It is easy to show that the stress tensor given in Example 4.1 satisfies the equilibrium equation. For this purpose, note first that in this particular case the forces $f$ per unit volume are $f = -\rho g e_2$, or in component form,

$$[f] = \begin{bmatrix} 0 \\ -\rho g \end{bmatrix}$$

Additionally, using Definition (2.134), the two-dimensional components of the divergence of $\sigma$ are,

$$[\operatorname{div} \sigma] = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \rho_0 g \frac{dX_2}{dx_2} \end{bmatrix}$$

which combined with the mass conservation equation $\rho dx_1 dx_2 = \rho_0 dX_1 dX_2$ and the lack of lateral deformation implies that Equation (4.14) is satisfied.

### 4.3.2 Rotational Equilibrium

Thus far the well-known symmetry of the Cauchy stresses has not been established. This is achieved by considering the rotational equilibrium of a general body, again under the action of traction and body forces. This implies that the total moment of body and traction forces about any arbitrary point, such as the origin, must vanish, that is,

$$\int_{\partial v} x \times t \, da + \int_v x \times f \, dv = 0 \tag{4.17}$$

where it should be recalled that the cross product of a force with a position vector $x$ yields the moment of that force about the origin. Equation (4.7a) for the traction vector in terms of the Cauchy stress tensor enables the above equation to be rewritten as,

$$\int_{\partial v} x \times (\sigma n)\, da + \int_v x \times f\, dv = 0 \tag{4.18}$$

Using the Gauss theorem and after some algebra, the equation becomes*

$$\int_v x \times (\operatorname{div} \sigma)\, dv + \int_v \mathcal{E} : \sigma^T\, dv + \int_v x \times f\, dv = \mathbf{0} \tag{4.19}$$

where $\mathcal{E}$ is the third-order alternating tensor, defined in Section 2.2.4 ($\mathcal{E}_{ijk} = 1$ if the permutation $\{i, j, k\}$ is even, $-1$ if it is odd, and zero if any indices are repeated.), so that the vector $\mathcal{E} : \sigma^T$ is,

$$\mathcal{E} : \sigma^T = \begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix} \tag{4.20}$$

Rearranging terms in Equation (4.19) to take into account the translational equilibrium Equation (4.15) and noting that the resulting equation is valid for any enclosed region of the body gives,

$$\mathcal{E} : \sigma^T = 0 \tag{4.21}$$

which, in view of Equation (4.20), clearly implies the symmetry of the Cauchy stress tensor $\sigma$.

## 4.4 PRINCIPLE OF VIRTUAL WORK

Generally, the finite element formulation is established in terms of a weak form of the differential equations under consideration. In the context of solid mechanics this implies the use of the virtual work equation. For this purpose, let $\delta v$ denote an arbitrary virtual velocity from the current position of the body as shown in Figure 4.6. The virtual work, $\delta w$, per unit volume and time done by the residual force $r$ during this virtual motion is $r \cdot \delta v$, and equilibrium implies,

$$\delta w = r \cdot \delta v = 0 \tag{4.22}$$

* To show this it is convenient to use indicial notation and the summation convention whereby repeated indices imply addition. Equation (2.136) then gives,

$$\begin{aligned} \int_{\partial v} \mathcal{E}_{ijk} x_j \sigma_{kl} n_l\, da &= \int_v \frac{\partial}{\partial x_l} (\mathcal{E}_{ijk} x_j \sigma_{kl})\, dv \\ &= \int_v \mathcal{E}_{ijk} x_j \frac{\partial \sigma_{kl}}{\partial x_l} + \int_v \mathcal{E}_{ijk} \sigma_{kj}\, dv \\ &= \int_v (x \times \operatorname{div} \sigma)_i\, dv + \int_v (\mathcal{E} : \sigma^T)_i\, dv \end{aligned}$$



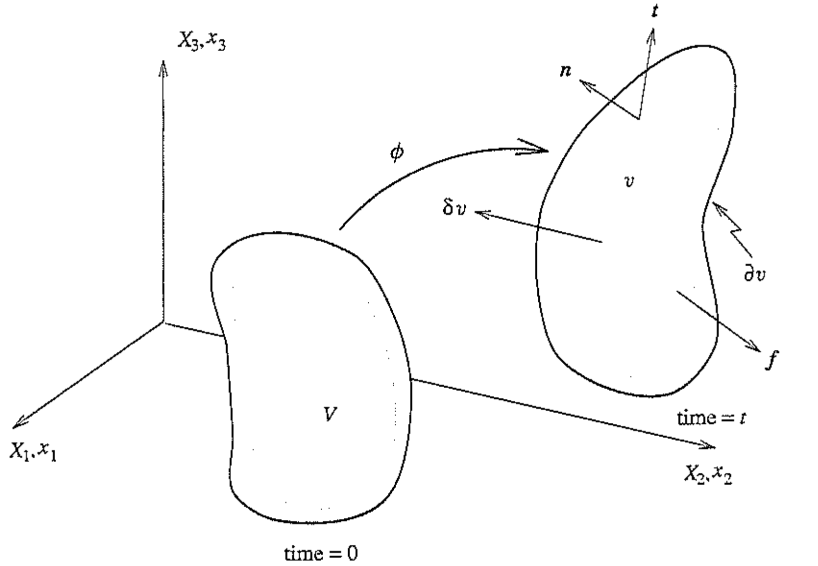



**FIGURE 4.6** Principle of virtual work.

Note that the above scalar equation is fully equivalent to the vector equation $r = \mathbf{0}$. This is due to the fact that $\delta v$ is arbitrary, and hence by choosing $\delta v = [1, 0, 0]^T$, followed by $\delta v = [0, 1, 0]^T$ and $\delta v = [0, 0, 1]^T$, the three components of the equation $r = \mathbf{0}$ are retrieved. We can now use Equation (4.16) for the residual vector and integrate over the volume of the body to give a weak statement of the static equilibrium of the body as,

$$\delta W = \int_v (\operatorname{div} \sigma + f) \cdot \delta v\, dv = 0 \tag{4.23}$$

A more common and useful expression can be derived by recalling Property (2.135e) to give the divergence of the vector $\sigma\delta v$ as,

$$\operatorname{div}(\sigma \delta v) = (\operatorname{div} \sigma) \cdot \delta v + \sigma : \nabla \delta v \tag{4.24}$$

Using this equation together with the Gauss theorem enables Equation (4.23) to be rewritten as,

$$\int_{\partial v} n \cdot \sigma \delta v\, da - \int_v \sigma : \nabla \delta v\, dv + \int_v f \cdot \delta v\, dv = 0 \tag{4.25}$$

The gradient of $\delta v$ is, by definition, the virtual velocity gradient $\delta l$. Additionally, we can use Equation (4.7a) for the traction vector and the symmetry of $\sigma$ to rewrite $\boldsymbol{n} \cdot \boldsymbol{\sigma}\delta\boldsymbol{v}$ as $\delta\boldsymbol{v} \cdot \boldsymbol{t}$, and consequently Equation (4.24) becomes,

$$\int_v \boldsymbol{\sigma} : \delta\boldsymbol{l}\, dv = \int_v \boldsymbol{f} \cdot \delta\boldsymbol{v}\, dv + \int_{\partial v} \boldsymbol{t} \cdot \delta\boldsymbol{v}\, da \tag{4.26}$$

Finally, expressing the virtual velocity gradient in terms of the symmetric virtual rate of deformation $\delta\boldsymbol{d}$ and the antisymmetric virtual spin tensor $\delta\boldsymbol{w}$ and taking into account again the symmetry of $\sigma$ gives the *spatial virtual work equation* as,

$$\delta W = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{d}\, dv - \int_v \boldsymbol{f} \cdot \delta\boldsymbol{v}\, dv - \int_{\partial v} \boldsymbol{t} \cdot \delta\boldsymbol{v}\, da = 0 \tag{4.27}$$

This fundamental scalar equation states the equilibrium of a deformable body and will become the basis for the finite element discretization.

## 4.5 WORK CONJUGACY AND ALTERNATIVE STRESS REPRESENTATIONS

### 4.5.1 The Kirchhoff Stress Tensor

In Equation (4.27) the internal virtual work done by the stresses is expressed as,

$$\delta W_{\text{int}} = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{d}\, dv \tag{4.28}$$

Pairs such as $\boldsymbol{\sigma}$ and $\boldsymbol{d}$ in this equation are said to be *work conjugate* with respect to the current deformed volume in the sense that their product gives work per unit current volume. Expressing the virtual work equation in the material coordinate system, alternative work conjugate pairs of stresses and strain rates will emerge. To achieve this objective, the spatial virtual work Equation (4.27) is first expressed with respect to the initial volume and area by transforming the integrals using Equation (3.56) for $dv$ to give,

$$\int_V J\boldsymbol{\sigma} : \delta\boldsymbol{d}\, dV = \int_V \boldsymbol{f}_0 \cdot \delta\boldsymbol{v}\, dV + \int_{\partial V} \boldsymbol{t}_0 \cdot \delta\boldsymbol{v}\, dA \tag{4.29}$$

where $\boldsymbol{f}_0 = J\boldsymbol{f}$ is the body force per unit undeformed volume and $\boldsymbol{t}_0 = \boldsymbol{t}(da/dA)$ is the traction vector per unit initial area, where the area ratio can be obtained after some algebra from Equation (3.68) as,

$$\frac{da}{dA} = \frac{J}{\sqrt{\boldsymbol{n} \cdot \boldsymbol{b}\boldsymbol{n}}} \tag{4.30}$$

The internal virtual work given by the left-hand side of Equation (4.29) can be expressed in terms of the *Kirchhoff stress tensor* $\boldsymbol{\tau}$ as,

$$\delta W_{\text{int}} = \int_V \boldsymbol{\tau} : \delta\boldsymbol{d}\, dV; \qquad \boldsymbol{\tau} = J\boldsymbol{\sigma} \tag{4.31a,b}$$



This equation reveals that the Kirchhoff stress tensor $\boldsymbol{\tau}$ is work conjugate to the rate of deformation tensor with respect to the initial volume. Note that the work per unit current volume is not equal to the work per unit initial volume. However, Equation (4.31b) and the relationship $\rho = \rho_0/J$ ensure that the work per unit mass is invariant and can be equally written in the current or initial configuration as:

$$\frac{1}{\rho}\boldsymbol{\sigma} : \boldsymbol{d} = \frac{1}{\rho_0}\boldsymbol{\tau} : \boldsymbol{d} \tag{4.32}$$

### 4.5.2 The First Piola–Kirchhoff Stress Tensor

The crude transformation that resulted in the internal virtual work given above is not entirely satisfactory because it still relies on the spatial quantities $\boldsymbol{\tau}$ and $\boldsymbol{d}$. To alleviate this lack of consistency, note that the symmetry of $\boldsymbol{\sigma}$ together with Equation (3.93) for $\boldsymbol{l}$ in terms of $\dot{\boldsymbol{F}}$ and the properties of the trace give,

$$\begin{aligned}
\delta W_{\text{int}} &= \int_V J\boldsymbol{\sigma} : \delta\boldsymbol{l}\, dV \\
&= \int_V J\boldsymbol{\sigma} : (\delta\dot{\boldsymbol{F}}\boldsymbol{F}^{-1})\, dV \\
&= \int_V \text{tr}(J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\delta\dot{\boldsymbol{F}})\, dV \\
&= \int_V (J\boldsymbol{\sigma}\boldsymbol{F}^{-T}) : \delta\dot{\boldsymbol{F}}\, dV
\end{aligned} \tag{4.33}$$

We observe from this equality that the stress tensor work conjugate to the rate of the deformation gradient $\dot{\boldsymbol{F}}$ is the so-called *first Piola–Kirchhoff stress tensor* given as,

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} \tag{4.34a}$$

Note that like $\boldsymbol{F}$, the first Piola–Kirchhoff tensor is an unsymmetric two-point tensor with components given as,

$$\boldsymbol{P} = \sum_{i,I=1}^{3} P_{iI}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_I; \qquad P_{iI} = \sum_{j=1}^{3} J\sigma_{ij}(\boldsymbol{F}^{-1})_{Ij} \tag{4.34b,c}$$

We can now rewrite the equation for the principle of virtual work in terms of the first Piola–Kirchhoff tensor as,

$$\int_V \boldsymbol{P} : \delta\dot{\boldsymbol{F}} dV = \int_V \boldsymbol{f}_0 \cdot \delta\boldsymbol{v} dV + \int_{\partial V} \boldsymbol{t}_0 \cdot \delta\boldsymbol{v} dA \tag{4.35}$$

Additionally, if the procedure employed to obtain the virtual work Equation (4.27) from the spatial differential equilibrium Equation (4.24) is reversed, an equivalent version of the differential equilibrium equation is obtained in terms of the first

Piola–Kirchhoff stress tensor as,

$$r_0 = Jr = \mathrm{DIV}\, P + f_0 = 0 \tag{4.36}$$

where DIV $P$ is the divergence of $P$ with respect to the initial coordinate system given as,

$$\mathrm{DIV}\, P = \nabla_0 P : I; \qquad \nabla_0 P = \frac{\partial P}{\partial X} \tag{4.37}$$

*Remark 1:* It is instructive to re-examine the physical meaning of the Cauchy stresses and thence the first Piola–Kirchhoff stress tensor. An element of force $dp$ acting on an element of area $da = n\, da$ in the spatial configuration can be written as,

$$dp = t da = \sigma da \tag{4.38}$$

Broadly speaking, the Cauchy stresses give the current force per unit deformed area, which is the familiar description of stress. Using Equation (3.68) for the spatial area vector, $dp$ can be rewritten in terms of the undeformed area corresponding to $da$ to give an expression involving the first Piola–Kirchhoff stresses as,

$$dp = J\sigma F^{-T} dA = P dA \tag{4.39}$$

This equation reveals that $P$, like $F$, is a two-point tensor that relates an area vector in the initial configuration to the corresponding force vector in the current configuration as shown in Figure 4.7. Consequently, the first Piola–Kirchhoff stresses can be loosely interpreted as the current force per unit of undeformed area.

### EXAMPLE 4.3: Rectangular block under self-weight (iii)

Using the physical interpretation for $P$ given in Remark 1 we can find the first Piola–Kirchhoff tensor corresponding to the state of stresses described in Example 4.1. For this purpose note first that dividing Equation (4.39) by the current area element $da$ gives the traction vector associated with a unit normal $N$ in the initial configuration as,

$$t(N) = PN\frac{dA}{da}$$

Using this equation with $N = E_2$ for the case described in Example 4.1 where the lack of lateral deformation implies $da = dA$ gives,

$$\begin{aligned} t(E_2) &= PE_2 \\ &= \sum_{i,I=1}^{2} P_{iI}(e_i \otimes E_I)E_2 \\ &= P_{12}e_1 + P_{22}e_2 \end{aligned}$$

*(continued)*



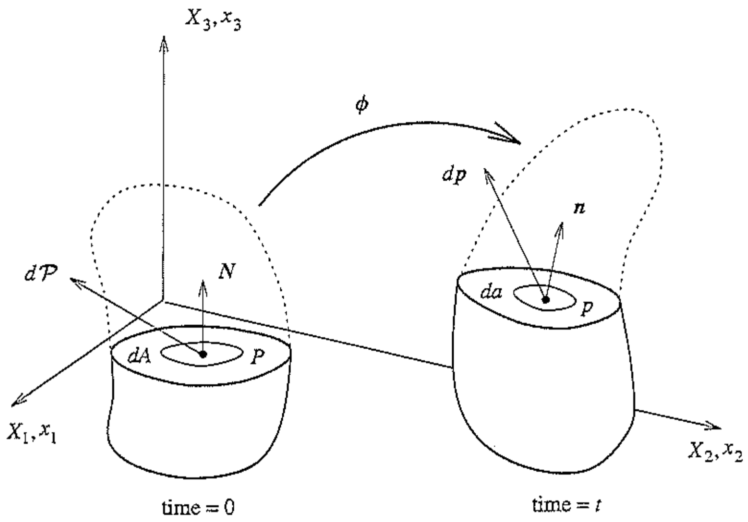



**FIGURE 4.7** Interpretation of stress tensors.

### EXAMPLE 4.3 *(cont.)*

Combining the final equation with the fact that $t(E_2) = t(e_2) = -\rho_0 g(H - X_2)e_2$ as explained in Example 4.1, we can identify $P_{12} = 0$ and $P_{22} = \rho_0 g(X_2 - H)$. Using a similar analysis for $t(E_1)$ eventually yields the components of $P$ as,

$$[P] = \begin{bmatrix} 0 & 0 \\ 0 & \rho_0 g(X_2 - H) \end{bmatrix}$$

which for this particular example coincide with the components of the Cauchy stress tensor. In order to show that the above tensor $P$ satisfies the equilibrium Equation (4.37), we first need to evaluate the force vector $f_0$ per unit initial volume as,

$$\begin{aligned} f_0 &= f\frac{dv}{dV} \\ &= -\rho\frac{dv}{dV}ge_2 \\ &= -\rho_0 g e_2 \end{aligned}$$

Combining this expression with the divergence of the above tensor $P$ immediately leads to the desired result.

## 4.5.3 The Second Piola–Kirchhoff Stress Tensor

The first Piola-Kirchhoff tensor $P$ is an unsymmetric two-point tensor and as such is not completely related to the material configuration. It is possible to contrive a totally material symmetric stress tensor, known as the second Piola–Kirchhoff stress $S$, by pulling back the spatial element of force $dp$ from Equation (4.39) to give a material force vector $d\mathcal{P}$ as,

$$d\mathcal{P} = \phi_*^{-1}[dp] = F^{-1}dp \tag{4.40}$$

Substituting from Equation (4.39) for $dp$ gives the transformed force in terms of the *second Piola–Kirchhoff stress tensor* $S$ and the material element of area $dA$ as,

$$d\mathcal{P} = S\,dA; \qquad S = JF^{-1}\sigma F^{-T} \tag{4.41a,b}$$

It is now necessary to derive the strain rate work conjugate to the second Piola–Kirchhoff stress in the following manner. From Equation (3.100) it follows that the material and spatial virtual rates of deformation are related as,

$$\delta d = F^{-T}\delta\dot{E}F^{-1} \tag{4.42}$$

Substituting this relationship into the internal virtual work Equation (4.28) gives,

$$\begin{aligned}\delta W_{\text{int}} &= \int_v \sigma : \delta d\, dv \\ &= \int_V J\sigma : (F^{-T}\delta\dot{E}F^{-1})\, dV \\ &= \int_V \text{tr}(F^{-1}J\sigma F^{-T}\delta\dot{E})\, dV \\ &= \int_V S : \delta\dot{E}\, dV\end{aligned} \tag{4.43}$$

which shows that $S$ is work conjugate to $\dot{E}$ and enables the *material virtual work equation* to be alternatively written in terms of the second Piola–Kirchhoff tensor as,

$$\int_V S : \delta\dot{E}\, dV = \int_V f_0 \cdot \delta v\, dV + \int_{\partial V} t_0 \cdot \delta v\, dA \tag{4.44}$$

For completeness the inverse of Equations (4.34a) and (4.41b) are given as,

$$\sigma = J^{-1}PF^T; \qquad \sigma = J^{-1}FSF^T \tag{4.45a,b}$$

*Remark 2:* Applying the pull back and push forward concepts to the Kirchhoff and second Piola–Kirchhoff tensors yields,

$$S = F^{-1}\tau F^{-T} = \phi_*^{-1}[\tau]; \qquad \tau = FSF^T = \phi_*[S] \tag{4.46a,b}$$

from which the second Piola–Kirchhoff and the Cauchy stresses are related as,

$$S = J\phi_*^{-1}[\sigma]; \qquad \sigma = J^{-1}\phi_*[S] \tag{4.47a,b}$$



In the above equation $S$ and $\sigma$ are related by the so-called *Piola transformation* which involves a push forward or pull back operation combined with the volume scaling $J$.

*Remark 3:* A useful interpretation of the second Piola–Kirchhoff stress can be obtained by observing that in the case of rigid body motion the polar decomposition given by Equation (3.27) indicates that $F = R$ and $J = 1$. Consequently, the second Piola–Kirchhoff stress tensor becomes,

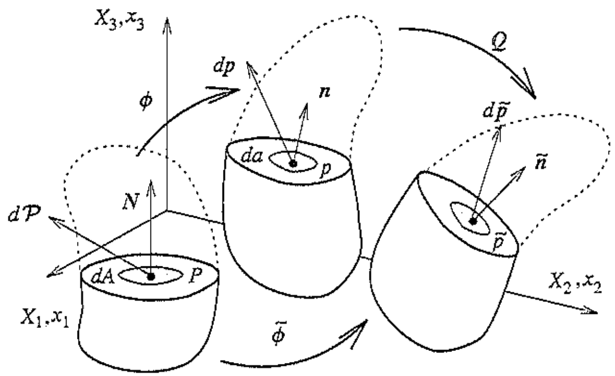
$$S = R^T\sigma R \tag{4.48}$$

Comparing this equation with the transformation Equations (2.42) given in Section 2.2.2, it transpires that the second Piola–Kirchhoff stress components coincide with the components of the Cauchy stress tensor expressed in the local set of orthogonal axes that results from rotating the global Cartesian directions according to $R$.

**EXAMPLE 4.4: Independence of $S$ from $Q$**

A useful property of the second Piola–Kirchhoff tensor $S$ is its independence from possible superimposed rotations $Q$ on the current body configuration. To prove this, note first that because $\tilde{\phi} = Q\phi$, then $\tilde{F} = QF$ and $\tilde{J} = J$. Using these equations in conjunction with the objectivity of $\sigma$ as given by Equation (4.11) gives,

$$\begin{aligned}\tilde{S} &= \tilde{J}\tilde{F}^{-1}\tilde{\sigma}\tilde{F}^{-T} \\ &= JF^{-1}Q^TQ\sigma Q^TQF^{-T} \\ &= S\end{aligned}$$



**EXAMPLE 4.5: Biot stress tensor**

Alternative stress tensors work conjugate to other strain measures can be contrived. For instance the material stress tensor $T$ work conjugate to the rate of the stretch tensor $\dot{U}$ is associated with the name of Biot. In order to derive a relationship between $T$ and $S$ note first that differentiating with respect to time the equations $UU = C$ and $2E = C - I$ gives,

$$\dot{E} = \tfrac{1}{2}(U\dot{U} + \dot{U}U)$$

With the help of this relationship we can express the internal work per unit of initial volume as,

$$\begin{aligned} S : \dot{E} &= S : \tfrac{1}{2}(U\dot{U} + \dot{U}U) \\ &= \tfrac{1}{2}\mathrm{tr}\,(SU\dot{U} + S\dot{U}U) \\ &= \tfrac{1}{2}\mathrm{tr}\,(SU\dot{U} + US\dot{U}) \\ &= \tfrac{1}{2}(SU + US) : \dot{U} \end{aligned}$$

and therefore the Biot tensor work conjugate to the stretch tensor is,

$$T = \tfrac{1}{2}(SU + US)$$

Using the polar decomposition and the relationship between $S$ and $P$, namely, $P = FS$, an alternative equation for $T$ emerges as,

$$T = \tfrac{1}{2}(R^TP + P^TR)$$

### 4.5.4 Deviatoric and Pressure Components

In many practical applications such as metal plasticity, soil mechanics, and biomechanics, it is physically relevant to isolate the hydrostatic pressure component $p$ from the deviatoric component $\sigma'$ of the Cauchy stress tensor as,

$$\sigma = \sigma' + pI; \qquad p = \tfrac{1}{3}\mathrm{tr}\,\sigma = \tfrac{1}{3}\sigma : I \tag{4.49a,b}$$

where the deviatoric Cauchy stress tensor $\sigma'$ satisfies $\mathrm{tr}\,\sigma' = 0$.

Similar decompositions can be established in terms of the first and second Piola–Kirchhoff stress tensors. For this purpose, we simply substitute the above decomposition into Equations (4.34a) for $P$ and (4.41b) for $S$ to give,

$$P = P' + pJF^{-T}; \qquad P' = J\sigma'F^{-T} \tag{4.50a}$$

$$S = S' + pJC^{-1}; \qquad S' = JF^{-1}\sigma'F^{-T} \tag{4.50b}$$

The tensors $S'$ and $P'$ are often referred to as the true deviatoric components of



$S$ and $P$. Note that although the trace of $\sigma'$ is zero, it does not follow that the traces of $S'$ and $P'$ must also vanish. In fact, the corresponding equations can be obtained from Equations (4.50a–b) and Properties (2.28,49) of the trace and double contractions as,

$$S' : C = 0 \tag{4.51a}$$

$$P' : F = 0 \tag{4.51b}$$

The above equations are important as they enable the hydrostatic pressure $p$ to be evaluated directly from either $S$ or $P$ as,

$$p = \tfrac{1}{3}J^{-1}P : F \tag{4.52a}$$

$$p = \tfrac{1}{3}J^{-1}S : C \tag{4.52b}$$

Proof of the above equations follows rapidly by taking the double contractions of (4.50a) by $F$ and (4.50b) by $C$.

**EXAMPLE 4.6: Proof of Equation (4.51a)**

Equation (4.51a) is easily proved as follows:

$$\begin{aligned} S' : C &= (JF^{-1}\sigma'F^{-T}) : C \\ &= J\mathrm{tr}\,(F^{-1}\sigma'F^{-T}C) \\ &= J\mathrm{tr}\,(\sigma'F^{-T}F^TFF^{-1}) \\ &= J\mathrm{tr}\,\sigma' \\ &= 0 \end{aligned}$$

A similar procedure can be used for (4.51b).

## 4.6 STRESS RATES

In Section 3.15 objective tensors were defined by imposing that under rigid body motions they transform according to Equation (3.135). Unfortunately, time differentiation of Equation (4.11) shows that the material time derivative of the stress tensor, $\dot{\sigma}$, fails to satisfy this condition as,

$$\dot{\tilde{\sigma}} = Q\dot{\sigma}Q^T + \dot{Q}\sigma Q^T + Q\sigma\dot{Q}^T \tag{4.53}$$

Consequently, $\dot{\tilde{\sigma}} \neq Q\dot{\sigma}Q^T$ unless the rigid body rotation is not a time-dependent transformation. Many rate-dependent materials, however, must be described in terms of stress rates and the resulting constitutive models must be frame-indifferent.

# Appendix

# F

## Exercises

1. A two-dimensional Cauchy stress tensor is given as,

$$\boldsymbol{\sigma} = \boldsymbol{t} \otimes \boldsymbol{n}_1 + \alpha \boldsymbol{n}_1 \otimes \boldsymbol{n}_2$$

where $\boldsymbol{t}$ is an arbitrary vector and $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are orthogonal unit vectors. (a) Describe graphically the state of stress. (b) Determine the value of $\alpha$ (hint: $\boldsymbol{\sigma}$ must be symmetric).
2. Using Equation (4.55) and a process similar to that employed in Example 4.5, show that, with respect to the initial volume, the stress tensor $\boldsymbol{\Pi}$ is work conjugate to the tensor $\dot{\boldsymbol{H}}$, where $\boldsymbol{H} = -\boldsymbol{F}^{-T}$ and $\boldsymbol{\Pi} = \boldsymbol{PC} = J\boldsymbol{\sigma F}$.
3. Using the time derivative of the equality $\boldsymbol{CC}^{-1} = \boldsymbol{I}$, show that the tensor $\boldsymbol{\Sigma} = \boldsymbol{CSC} = J\boldsymbol{F}^T\boldsymbol{\sigma F}$ is work conjugate to $\frac{1}{2}\dot{\boldsymbol{B}}$, where $\boldsymbol{B} = -\boldsymbol{C}^{-1}$. Find relationships between $\boldsymbol{T}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{\Pi}$.
4. Prove Equation (4.51b) $\boldsymbol{P}' : \boldsymbol{F} = 0$ using a procedure similar to Example 4.6.
5. Prove directly that the Jaumann stress tensor, $\boldsymbol{\sigma}^{\nabla}$ is an objective tensor, using a procedure similar to Example 4.7.
6. Prove that if $d\boldsymbol{x}_1$ and $d\boldsymbol{x}_2$ are two arbitrary elemental vectors moving with the body (see Figure 3.2) then:

$$\frac{d}{dt}(d\boldsymbol{x}_1 \cdot \boldsymbol{\sigma} d\boldsymbol{x}_2) = d\boldsymbol{x}_1 \cdot \boldsymbol{\sigma}^{\diamond} d\boldsymbol{x}_2$$

CHAPTER FIVE

# HYPERELASTICITY

## 5.1 INTRODUCTION

The equilibrium equations derived in the previous section are written in terms of the stresses inside the body. These stresses result from the deformation of the material, and it is now necessary to express them in terms of some measure of this deformation such as, for instance, the strain. These relationships, known as constitutive equations, obviously depend on the type of material under consideration and may be dependent on or independent of time. For example the classical small strain linear elasticity equations involving Young modulus and Poisson ratio are time-independent, whereas viscous fluids are clearly entirely dependent on strain rate.

Generally, constitutive equations must satisfy certain physical principles. For example, the equations must obviously be objective, that is, frame-invariant. In this chapter the constitutive equations will be established in the context of a hyperelastic material, whereby stresses are derived from a stored elastic energy function. Although there are a number of alternative material descriptions that could be introduced, hyperelasticity is a particularly convenient constitutive equation given its simplicity and that it constitutes the basis for more complex material models such as elastoplasticity, viscoplasticity, and viscoelasticity.

## 5.2 HYPERELASTICITY

Materials for which the constitutive behavior is only a function of the current state of deformation are generally known as *elastic*. Under such conditions, any stress measure at a particle $X$ is a function of the current deformation gradient $\boldsymbol{F}$ associated with that particle. Instead of using any of the alternative strain measures given in Chapter 3, the deformation gradient $\boldsymbol{F}$, together with its conjugate first

Piola–Kirchhoff stress measure $P$, will be retained in order to define the basic material relationships. Consequently, elasticity can be generally expressed as,

$$P = P(F(X), X) \tag{5.1}$$

where the direct dependency upon $X$ allows for the possible inhomogeneity of the material.

In the special case when the work done by the stresses during a deformation process is dependent only on the initial state at time $t_0$ and the final configuration at time $t$, the behavior of the material is said to be path-independent and the material is termed *hyperelastic*. As a consequence of the path-independent behavior and recalling from Equation (4.31) that $P$ is work conjugate with the rate of deformation gradient $\dot{F}$, a *stored strain energy function* or *elastic potential* $\Psi$ per unit undeformed volume can be established as the work done by the stresses from the initial to the current position as,

$$\Psi(F(X), X) = \int_{t_0}^{t} P(F(X), X) : \dot{F}\, dt; \qquad \dot{\Psi} = P : \dot{F} \tag{5.2a,b}$$

Presuming that from physical experiments it is possible to construct the function $\Psi(F, X)$, which defines a given material, then the rate of change of the potential can be alternatively expressed as,

$$\dot{\Psi} = \sum_{i,J=1}^{3} \frac{\partial \Psi}{\partial F_{iJ}} \dot{F}_{iJ} \tag{5.3}$$

Comparing this with Equation (5.2b) reveals that the components of the two-point tensor $P$ are,

$$P_{iJ} = \frac{\partial \Psi}{\partial F_{iJ}} \tag{5.4}$$

For notational convenience this expression is rewritten in a more compact form as,

$$P(F(X), X) = \frac{\partial \Psi(F(X), X)}{\partial F} \tag{5.5}$$

Equation (5.5) followed by Equation (5.2) is often used as a definition of a hyperelastic material.

The general constitutive Equation (5.5) can be further developed by recalling the restrictions imposed by objectivity as discussed in Section 3.15. To this end, $\Psi$ must remain invariant when the current configuration undergoes a rigid body rotation. This implies that $\Psi$ depends on $F$ only via the stretch component $U$ and is independent of the rotation component $R$. For convenience, however, $\Psi$ is often expressed as a function of $C = U^2 = F^T F$ as,

$$\Psi(F(X), X) = \Psi(C(X), X) \tag{5.6}$$



Observing that $\frac{1}{2}\dot{C} = \dot{E}$ is work conjugate to the second Piola–Kirchhoff stress $S$, enables a totally Lagrangian constitutive equation to be constructed in the same manner as Equation (5.5) to give,

$$\dot{\Psi} = \frac{\partial \Psi}{\partial C} : \dot{C} = \frac{1}{2} S : \dot{C}; \qquad S(C(X), X) = 2\frac{\partial \Psi}{\partial C} = \frac{\partial \Psi}{\partial E} \tag{5.7a,b}$$

## 5.3 ELASTICITY TENSOR

### 5.3.1 The Material or Lagrangian Elasticity Tensor

The relationship between $S$ and $C$ or $E = \frac{1}{2}(C - I)$, given by Equation (5.7b) will invariably be nonlinear. Within the framework of a potential Newton–Raphson solution process, this relationship will need to be linearized with respect to an increment $u$ in the current configuration. Using the chain rule, a linear relationship between the directional derivative of $S$ and the linearized strain $DE[u]$ can be obtained, initially in a component form, as,

$$\begin{aligned} DS_{IJ}[u] &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S_{IJ}(E_{KL}[\phi + \epsilon u]) \\ &= \sum_{K,L=1}^{3} \frac{\partial S_{IJ}}{\partial E_{KL}} \frac{d}{d\epsilon}\bigg|_{\epsilon=0} E_{KL}[\phi + \epsilon u] \\ &= \sum_{K,L=1}^{3} \frac{\partial S_{IJ}}{\partial E_{KL}} DE_{KL}[u] \end{aligned} \tag{5.8}$$

This relationship between the directional derivatives of $S$ and $E$ is more concisely expressed as,

$$DS[u] = \mathcal{C} : DE[u] \tag{5.9}$$

where the symmetric fourth-order tensor $\mathcal{C}$, known as the *Lagrangian* or *material elasticity tensor*, is defined by the partial derivatives as,

$$\begin{aligned} \mathcal{C} &= \sum_{I,J,K,L=1}^{3} \mathcal{C}_{IJKL}\, E_I \otimes E_J \otimes E_K \otimes E_L; \\ \mathcal{C}_{IJKL} &= \frac{\partial S_{IJ}}{\partial E_{KL}} = \frac{4\,\partial^2 \Psi}{\partial C_{IJ} \partial C_{KL}} = \mathcal{C}_{KLIJ} \end{aligned} \tag{5.10}$$

For convenience these expressions are often abbreviated as,

$$\boldsymbol{\mathcal{C}} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} = 2\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} = \frac{4\,\partial^2 \Psi}{\partial \boldsymbol{C}\partial \boldsymbol{C}} \tag{5.11}$$

**EXAMPLE 5.1: St. Venant–Kirchhoff Material**

The simplest example of a hyperelastic material is the St. Venant–Kirchhoff model, which is defined by a strain energy function $\Psi$ as,

$$\Psi(\boldsymbol{E}) = \frac{1}{2}\lambda(\operatorname{tr}\boldsymbol{E})^2 + \mu\boldsymbol{E} : \boldsymbol{E}$$

where $\lambda$ and $\mu$ are material coefficients. Using the second part of Equation (5.7b), we can obtain the second Piola–Kirchhoff stress tensor as,

$$\boldsymbol{S} = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}$$

and using Equation (5.10), the coefficients of the Lagrangian elasticity tensor emerge as,

$$\mathcal{C}_{IJKL} = \lambda\delta_{IJ}\delta_{KL} + 2\mu\delta_{IK}\delta_{JL}$$

Note that these two last equations are analogous to those used in linear elasticity, where the small strain tensor has been replaced by the Green strain. Unfortunately, this St. Venant–Kirchhoff material has been found to be of little practical use beyond the small strain regime.

### 5.3.2 The Spatial or Eulerian Elasticity Tensor

It would now be pertinent to attempt to find a spatial equivalent to Equation (5.9), and it would be tempting to suppose that this would involve a relationship between the linearized Cauchy stress and the linearized Almansi strain. Although, in principle, this can be achieved, the resulting expression is intractable. An easier route is to interpret Equation (5.9) in a rate form and apply the push forward operation to the resulting equation. This is achieved by linearizing $\boldsymbol{S}$ and $\boldsymbol{E}$ in the direction of $\boldsymbol{v}$, rather than $\boldsymbol{u}$. Recalling from Section 3.9.3 that $D\boldsymbol{S}[\boldsymbol{v}] = \dot{\boldsymbol{S}}$ and $D\boldsymbol{E}[\boldsymbol{v}] = \dot{\boldsymbol{E}}$ gives,

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{C}} : \dot{\boldsymbol{E}} \tag{5.12}$$

Because the push forward of $\dot{\boldsymbol{S}}$ has been shown in Section 4.5 to be the Truesdell rate of the Kirchhoff stress $\boldsymbol{\tau}^{\circ} = J\boldsymbol{\sigma}^{\circ}$ and the push forward of $\dot{\boldsymbol{E}}$ is $\boldsymbol{d}$, namely, Equation (3.91a), it is now possible to obtain the spatial equivalent of the material linearized constitutive Equation (5.12) as,

$$\boldsymbol{\sigma}^{\circ} = \mathbf{c} : \boldsymbol{d} \tag{5.13}$$

where $\mathbf{c}$, the *Eulerian or spatial elasticity tensor*, is defined as the Piola push



forward of $\boldsymbol{\mathcal{C}}$ and after some careful indicial manipulations can be obtained as*,

$$\mathbf{c} = J^{-1}\phi_*[\boldsymbol{\mathcal{C}}]; \qquad \mathbf{c} = \sum_{\substack{i,j,k,l=1\\ I,J,K,L=1}}^{3} J^{-1}F_{iI}F_{jJ}F_{kK}F_{lL}\mathcal{C}_{IJKL}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l \tag{5.14}$$

Often, Equation (5.13) is used, together with convenient coefficients in $\mathbf{c}$, as the fundamental constitutive equation that defines the material behavior. Use of such an approach will, in general, not guarantee hyperelastic behavior, and therefore the stresses cannot be obtained directly from an elastic potential. In such cases, the rate equation has to be integrated in time, and this can cause substantial difficulties in a finite element analysis because of problems associated with objectivity over a finite time increment.

*Remark 1:* Using Equations (3.96) and (4.55), it can be observed that Equation (5.13) can be reinterpreted in terms of Lie derivatives as,

$$\mathcal{L}_{\phi}[\boldsymbol{\tau}] = J\mathbf{c} : \mathcal{L}_{\phi}[\boldsymbol{e}] \tag{5.15}$$

## 5.4 ISOTROPIC HYPERELASTICITY

### 5.4.1 Material Description

The hyperelastic constitutive equations discussed so far are unrestricted in their application. We are now going to restrict these equations to the common and important isotropic case. Isotropy is defined by requiring the constitutive behavior to be identical in any material direction**. This implies that the relationship between $\Psi$ and $\boldsymbol{C}$ must be independent of the material axes chosen and, consequently, $\Psi$ must only be a function of the invariants of $\boldsymbol{C}$ as,

$$\Psi(\boldsymbol{C}(\boldsymbol{X}), \boldsymbol{X}) = \Psi(I_C, II_C, III_C, \boldsymbol{X}) \tag{5.16}$$

where the invariants of $\boldsymbol{C}$ are defined here as,

$$I_C = \operatorname{tr}\boldsymbol{C} = \boldsymbol{C} : \boldsymbol{I} \tag{5.17a}$$

$$II_C = \operatorname{tr}\boldsymbol{C}\boldsymbol{C} = \boldsymbol{C} : \boldsymbol{C} \tag{5.17b}$$

$$III_C = \det\boldsymbol{C} = J^2 \tag{5.17c}$$

* Using the standard summation convention and noting from Equation (4.54) that $\sigma^{\circ}_{ij} = J^{-1}F_{iI}F_{jJ}\dot{S}_{IJ}$ and from Equation (3.91a) that $\dot{E}_{KL} = F_{kK}F_{lL}d_{kl}$ gives,

$$\sigma^{\circ}_{ij} = J^{-1}\tau^{\circ}_{ij} = J^{-1}F_{iI}F_{jJ}\mathcal{C}_{IJKL}F_{kK}F_{lL}d_{kl} = \mathbf{c}_{ijkl}d_{kl}$$

and, consequently, $\mathbf{c}_{ijkl} = J^{-1}F_{iI}F_{jJ}F_{kK}F_{lL}\mathcal{C}_{IJKL}$.

** Note that the resulting spatial behavior as given by the spatial elasticity tensor may be anisotropic.

As a result of the isotropic restriction, the second Piola–Kirchhoff stress tensor can be rewritten from Equation (5.7b) as,

$$S = 2\frac{\partial \Psi}{\partial C} = 2\frac{\partial \Psi}{\partial I_C}\frac{\partial I_C}{\partial C} + 2\frac{\partial \Psi}{\partial II_C}\frac{\partial II_C}{\partial C} + 2\frac{\partial \Psi}{\partial III_C}\frac{\partial III_C}{\partial C} \tag{5.18}$$

The second-order tensors formed by the derivatives of the first two invariants with respect to $C$ can be evaluated in component form to give,

$$\frac{\partial}{\partial C_{IJ}}\sum_{K=1}^{3} C_{KK} = \delta_{IJ}; \qquad \frac{\partial I_C}{\partial C} = I \tag{5.19a}$$

$$\frac{\partial}{\partial C_{IJ}}\sum_{K,L=1}^{3} C_{KL}C_{KL} = 2C_{IJ}; \qquad \frac{\partial II_C}{\partial C} = 2C \tag{5.19b}$$

The derivative of the third invariant is more conveniently evaluated using the expression for the linearization of the determinant of a tensor given in Equation (2.119). To this end note that the directional derivative with respect to an arbitrary increment tensor $\Delta C$ and the partial derivatives are related via,

$$DIII_C[\Delta C] = \sum_{I,J=1}^{3} \frac{\partial III_C}{\partial C_{IJ}}\Delta C_{IJ} = \frac{\partial III_C}{\partial C} : \Delta C \tag{5.20}$$

Rewriting Equation (2.119) as,

$$DIII_C[\Delta C] = \det C\,(C^{-1} : \Delta C) \tag{5.21}$$

and comparing this equation with Expression (5.20) and noting that both equations are valid for any increment $\Delta C$ yields,

$$\frac{\partial III_C}{\partial C} = J^2 C^{-1} \tag{5.22}$$

Introducing Expressions (5.19a,b) and (5.22) into Equation (5.18) enables the second Piola–Kirchhoff stress to be evaluated as,

$$S = 2\Psi_I I + 4\Psi_{II} C + 2J^2\Psi_{III} C^{-1} \tag{5.23}$$

where $\Psi_I = \partial\Psi/\partial I_C$, $\Psi_{II} = \partial\Psi/\partial II_C$, and $\Psi_{III} = \partial\Psi/\partial III_C$.

### 5.4.2 Spatial Description

In design practice it is obviously the Cauchy stresses that are of engineering significance. These can be obtained indirectly from the second Piola–Kirchhoff stresses by using Equation (4.45b) as,

$$\sigma = J^{-1} F S F^T \tag{5.24}$$



Substituting $S$ from Equation (5.23) and noting that the left Cauchy–Green tensor is $b = FF^T$ gives,

$$\sigma = 2J^{-1}\Psi_I b + 4J^{-1}\Psi_{II} b^2 + 2J\Psi_{III} I \tag{5.25}$$

In this equation $\Psi_I$, $\Psi_{II}$, and $\Psi_{III}$ still involve derivatives with respect to the invariants of the material tensor $C$. Nevertheless it is easy to show that the invariants of $b$ are identical to the invariants of $C$, as the following expressions demonstrate,

$$I_b = \mathrm{tr}[b] = \mathrm{tr}[FF^T] = \mathrm{tr}[F^T F] = \mathrm{tr}[C] = I_C \tag{5.26a}$$

$$II_b = \mathrm{tr}[bb] = \mathrm{tr}[FF^T FF^T] = \mathrm{tr}[F^T F F^T F] = \mathrm{tr}[CC] = II_C \tag{5.26b}$$

$$III_b = \det[b] = \det[FF^T] = \det[F^T F] = \det[C] = III_C \tag{5.26c}$$

Consequently, the terms $\Psi_I$, $\Psi_{II}$, and $\Psi_{III}$ in Equation (5.25) are also the derivatives of $\Psi$ with respect to the invariants of $b$.

*Remark 2:* Note that any spatially based expression for $\Psi$ must be a function of $b$ only via its invariants, which implies an isotropic material. This follows from the condition that $\Psi$ must remain constant under rigid body rotations and only the invariants of $b$, not $b$ itself, remain unchanged under such rotations.

**EXAMPLE 5.2: Cauchy stresses**

It is possible to derive an alternative equation for the Cauchy stresses directly from the strain energy. For this purpose, note first that the time derivative of $b$ is,

$$\dot{b} = \dot{F}F^T + F\dot{F}^T = lb + bl^T$$

and therefore the internal energy rate per unit of undeformed volume $\dot{w}_0 = \dot{\Psi}$ is,

$$\begin{aligned}\dot{\Psi} &= \frac{\partial \Psi}{\partial b} : \dot{b} \\ &= \frac{\partial \Psi}{\partial b} : (lb + bl^T) \\ &= 2\frac{\partial \Psi}{\partial b}b : l\end{aligned}$$

If we combine this equation with the fact that $\sigma$ is work conjugate to $l$ with respect to the current volume, that is, $\dot{w} = J^{-1}\dot{w}_0 = \sigma : l$, gives,

$$J\sigma = 2\frac{\partial \Psi}{\partial b}b$$

It is simple to show that this equation gives the same result as Equation (5.25) for isotropic materials where $\Psi$ is a function of the invariants of $b$.

### 5.4.3 Compressible Neo-Hookean Material

The equations derived in the previous sections refer to a general isotropic hyperelastic material. We can now focus on a particularly simple case known as *compressible neo-Hookean* material. This material exhibits characteristics that can be identified with the familiar material parameters found in linear elastic analysis. The energy function of such a material is defined as,

$$\Psi = \frac{\mu}{2}(I_C - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2 \tag{5.27}$$

where the constants $\lambda$ and $\mu$ are material coefficients and $J^2 = III_C$. Note that in the absence of deformation, that is, when $\boldsymbol{C} = \boldsymbol{I}$, the stored energy function vanishes as expected.

The second Piola–Kirchhoff stress tensor can now be obtained from Equation (5.23) as,

$$\boldsymbol{S} = \mu(\boldsymbol{I} - \boldsymbol{C}^{-1}) + \lambda(\ln J)\boldsymbol{C}^{-1} \tag{5.28}$$

Alternatively, the Cauchy stresses can be obtained using Equation (5.25) in terms of the left Cauchy–Green tensor $\boldsymbol{b}$ as,

$$\boldsymbol{\sigma} = \frac{\mu}{J}(\boldsymbol{b} - \boldsymbol{I}) + \frac{\lambda}{J}(\ln J)\boldsymbol{I} \tag{5.29}$$

The Lagrangian elasticity tensor corresponding to this neo-Hookean material can be obtained by differentiation of Equation (5.28) with respect to the components of $\boldsymbol{C}$ to give, after some algebra using Equation (5.22), $\boldsymbol{\mathcal{C}}$ as,

$$\boldsymbol{\mathcal{C}} = \lambda \boldsymbol{C}^{-1} \otimes \boldsymbol{C}^{-1} + 2(\mu - \lambda \ln J)\boldsymbol{\mathcal{I}} \tag{5.30}$$

where $\boldsymbol{C}^{-1} \otimes \boldsymbol{C}^{-1} = \sum (C^{-1})_{IJ}(C^{-1})_{KL} \boldsymbol{E}_I \otimes \boldsymbol{E}_J \otimes \boldsymbol{E}_K \otimes \boldsymbol{E}_L$ and the fourth-order tensor $\boldsymbol{\mathcal{I}}$ is defined as,

$$\boldsymbol{\mathcal{I}} = -\frac{\partial \boldsymbol{C}^{-1}}{\partial \boldsymbol{C}}; \qquad \mathcal{I}_{IJKL} = -\frac{\partial (C^{-1})_{IJ}}{\partial C_{KL}} \tag{5.31}$$

In order to obtain the coefficients of this tensor, recall from Section 2.3.4 that the directional derivative of the inverse of a tensor in the direction of an arbitrary increment $\Delta \boldsymbol{C}$ is,

$$D\boldsymbol{C}^{-1}[\Delta \boldsymbol{C}] = -\boldsymbol{C}^{-1}(\Delta \boldsymbol{C})\boldsymbol{C}^{-1} \tag{5.32}$$

Alternatively, this directional derivative can be expressed in terms of the partial derivatives as,

$$D\boldsymbol{C}^{-1}[\Delta \boldsymbol{C}] = \frac{\partial \boldsymbol{C}^{-1}}{\partial \boldsymbol{C}} : \Delta \boldsymbol{C} \tag{5.33}$$



Consequently, the components of $\boldsymbol{\mathcal{I}}$ can be identified as,

$$\mathcal{I}_{IJKL} = (C^{-1})_{IK}(C^{-1})_{JL} \tag{5.34}$$

The Eulerian or spatial elasticity tensor can now be obtained by pushing forward the Lagrangian tensor using Equation (5.14) to give, after tedious algebra, $\boldsymbol{c}$ as,

$$\boldsymbol{c} = \frac{\lambda}{J}\boldsymbol{I} \otimes \boldsymbol{I} + \frac{2}{J}(\mu - \lambda \ln J)\boldsymbol{i} \tag{5.35}$$

where $\boldsymbol{i}$ is the fourth-order identity tensor obtained by pushing forward $\boldsymbol{\mathcal{I}}$ and in component form is given in terms of the Kroneker delta as,

$$\boldsymbol{i} = \phi_*[\boldsymbol{\mathcal{I}}]; \qquad i_{ijkl} = \sum_{I,J,K,L} F_{iI}F_{jJ}F_{kK}F_{lL}\mathcal{I}_{IJKL} = \delta_{ik}\delta_{jl} \tag{5.36}$$

Note that Equation (5.36) defines an isotropic fourth-order tensor as discussed in Section 2.2.4, similar to that used in linear elasticity, which can be expressed in terms of the effective Lamé moduli $\lambda'$ and $\mu'$ as,

$$c_{ijkl} = \lambda'\delta_{ij}\delta_{kl} + 2\mu'\delta_{ik}\delta_{jl} \tag{5.37}$$

where the effective coefficients $\lambda'$ and $\mu'$ are,

$$\lambda' = \frac{\lambda}{J}; \qquad \mu' = \frac{\mu - \lambda \ln J}{J} \tag{5.38}$$

Note that in the case of small strains when $J \approx 1$, then $\lambda' \approx \lambda$, $\mu' \approx \mu$, and the standard fourth-order tensor used in linear elastic analysis is recovered.

**EXAMPLE 5.3: Pure dilatation (i)**

The simplest possible deformation is a pure dilatation case where the deformation gradient tensor $\boldsymbol{F}$ is,

$$\boldsymbol{F} = \lambda \boldsymbol{I}; \qquad J = \lambda^3$$

and the left Cauchy–Green tensor $\boldsymbol{b}$ is therefore,

$$\boldsymbol{b} = \lambda^2 \boldsymbol{I} = J^{2/3}\boldsymbol{I}$$

Under such conditions the Cauchy stress tensor for a compressible neo-Hookean material is evaluated with the help of Equation (5.29) as,

$$\boldsymbol{\sigma} = \left[\frac{\mu}{J}(J^{2/3} - 1) + \frac{\lambda}{J}\ln J\right]\boldsymbol{I}$$

which represents a state of hydrostatic stress with pressure $p$ equal to,

$$p = \frac{\mu}{J}(J^{2/3} - 1) + \frac{\lambda}{J}\ln J$$

**EXAMPLE 5.4: Simple shear (i)**

The case of simple shear described in Chapter 3 is defined by a deformation gradient and left Cauchy–Green tensors as,

$$F = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad b = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which imply $J = 1$ and the Cauchy stresses for a neo-Hookean material are,

$$\sigma = \mu \begin{bmatrix} \gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that only when $\gamma \to 0$ is a state of pure shear obtained. Note also that despite the fact that $J = 1$, that is, there is no change in volume, the pressure $p = \mathrm{tr}\,\sigma/3 = \gamma^2/3$ is not zero. This is known as the *Kelvin* effect.

## 5.5 INCOMPRESSIBLE AND NEARLY INCOMPRESSIBLE MATERIALS

Most practical large strain processes take place under incompressible or near incompressible conditions. Hence it is pertinent to discuss the constitutive implications of this constraint on the deformation. The terminology "near incompressibility" is used here to denote materials that are truly incompressible, but their numerical treatment invokes a small measure of volumetric deformation. Alternatively, in a large strain elastoplastic or inelastic context, the plastic deformation is often truly incompressible and the elastic volumetric strain is comparatively small.

### 5.5.1 Incompressible Elasticity

In order to determine the constitutive equation for an incompressible hyperelastic material, recall Equation (5.7a) rearranged as:

$$\left(\frac{1}{2}S - \frac{\partial \Psi}{\partial C}\right) : \dot{C} = 0 \tag{5.39}$$

Previously the fact that $\dot{C}$ in this equation was arbitrary implied that $S = 2\partial\Psi/\partial C$. In the incompressible case, the term in brackets is not guaranteed to vanish because $\dot{C}$ is no longer arbitrary. In fact, given that $J = 1$ throughout the deformation and



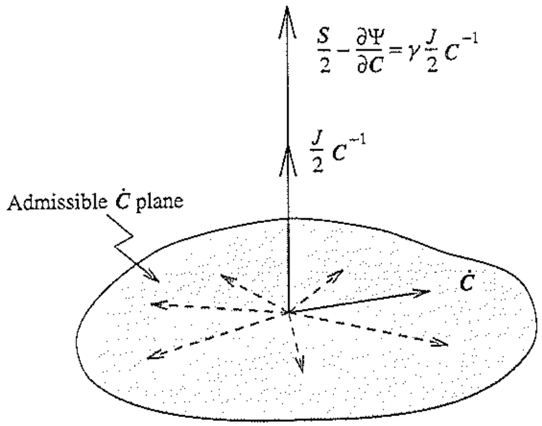



**FIGURE 5.1** Incompressibility constraint

therefore $\dot{J} = 0$, Equation (3.129) gives the required constraint on $\dot{C}$ as,

$$\tfrac{1}{2}JC^{-1} : \dot{C} = 0 \tag{5.40}$$

The fact that Equation (5.39) has to be satisfied for any $\dot{C}$ that complies with condition (5.40) implies that,

$$\frac{1}{2}S - \frac{\partial \Psi}{\partial C} = \gamma \frac{J}{2} C^{-1} \tag{5.41}$$

where $\gamma$ is an unknown scalar that will, under certain circumstances that we will discuss later, coincide with the hydrostatic pressure and will be determined by using the additional equation given by the incompressibility constraint $J = 1$. Equation (5.40) is symbolically illustrated in Figure 5.1, where the double contraction " : " has been interpreted as a generalized dot product. This enables $(S/2 - \partial\Psi/\partial C)$ and $JC^{-1}/2$ to be seen as being orthogonal to any admissible $\dot{C}$ and therefore proportional to each other.

From Equation (5.41) the general incompressible hyperelastic constitutive equation emerges as,

$$S = 2\frac{\partial \Psi(C)}{\partial C} + \gamma J C^{-1} \tag{5.42}$$

The determinant $J$ in the above equation may seem unnecessary in the case of incompressibility where $J = 1$, but retaining $J$ has the advantage that Equation (5.42) is also applicable in the nearly incompressible case. Furthermore, in practical terms, a finite element analysis rarely enforces $J = 1$ in a strict pointwise manner, and hence its retention may be important for the evaluation of stresses.

Recalling Equation (4.50b) giving the deviatoric–hydrostatic decomposition of the second Piola–Kirchhoff tensor as $S = S' + pJC^{-1}$, it would be convenient to identify the parameter $\gamma$ with the pressure $p$. With this in mind, a relationship between $p$ and $\gamma$ can be established to give,

$$\begin{aligned} p &= \frac{1}{3}J^{-1}S : C \\ &= \frac{1}{3}J^{-1}\left[2\frac{\partial \Psi}{\partial C} + \gamma J C^{-1}\right] : C \\ &= \gamma + \frac{2}{3}J^{-1}\frac{\partial \Psi}{\partial C} : C \end{aligned} \tag{5.43}$$

which clearly indicates that $\gamma$ and $p$ coincide only if,

$$\frac{\partial \Psi}{\partial C} : C = 0 \tag{5.44}$$

This implies that the function $\Psi(C)$ must be homogeneous of order 0, that is, $\Psi(\alpha C) = \Psi(C)$ for any arbitrary constant $\alpha$.* This can be achieved by recognizing that for incompressible materials $III_C = \det C = J^2 = 1$. We can therefore express the energy function $\Psi$ in terms of the distortional component of the right Cauchy–Green tensor $\hat{C} = III_C^{-1/3}C$ to give a formally modified energy function $\widehat{\Psi}(C) = \Psi(\hat{C})$. The homogeneous properties of the resulting function $\widehat{\Psi}(C)$ are easily shown by,

$$\begin{aligned} \widehat{\Psi}(\alpha C) &= \Psi[(\det \alpha C)^{-1/3}(\alpha C)] \\ &= \Psi[(\alpha^3 \det C)^{-1/3}\alpha C] \\ &= \Psi[(\det C)^{-1/3}C] \\ &= \widehat{\Psi}(C) \end{aligned} \tag{5.45}$$

Accepting that for the case of incompressible materials $\Psi$ can be replaced by $\widehat{\Psi}$, Condition (5.44) is satisfied and Equation (5.42) becomes,

$$S = 2\frac{\partial \widehat{\Psi}(C)}{\partial C} + pJC^{-1} \tag{5.46}$$

It is now a trivial matter to identify the deviatoric component of the second Piola–Kirchhoff tensor by comparison of the above equation with Equation (4.50b) to give,

$$S' = 2\frac{\partial \widehat{\Psi}}{\partial C} \tag{5.47}$$

Note that the derivative $\partial\widehat{\Psi}(C)/\partial C$ is not equal to the derivative $\partial\Psi(C)/\partial C$, despite the fact that $\hat{C} = C$ for incompressibility. This is because $III_C$ remains a function of $C$ while the derivative of $\hat{C}$ is being executed. It is only after the derivative has been completed that the substitution $III_C = 1$ can be made.

* A scalar function $f(x)$ of a $k$-dimensional vector variable $x = [x_1, x_2, ..., x_k]^T$ is said to be homogeneous of order $n$ if for any arbitrary constant $\alpha$,

$$f(\alpha x) = \alpha^n f(x)$$

Differentiating this expression with respect to $\alpha$ at $\alpha = 1$ gives,

$$\frac{\partial f}{\partial x} \cdot x = nf(x)$$



### 5.5.2 Incompressible Neo-Hookean Material

In the case of incompressibility the neo-Hookean material introduced in Section 5.4.3 is defined by a hyperelastic potential $\Psi(C)$ given as,

$$\Psi(C) = \frac{1}{2}\mu(\operatorname{tr} C - 3) \tag{5.48}$$

The equivalent homogeneous potential $\widehat{\Psi}$ is established by replacing $C$ by $\hat{C}$ to give,

$$\widehat{\Psi}(C) = \frac{1}{2}\mu(\operatorname{tr} \hat{C} - 3) \tag{5.49}$$

Now using Equation (5.46) $S$ is obtained with the help of Equations (5.19a) and (5.20) as,

$$\begin{aligned} S &= 2\frac{\partial \widehat{\Psi}(C)}{\partial C} + pJC^{-1} \\ &= \mu\frac{\partial \operatorname{tr}\hat{C}}{\partial C} + pJC^{-1} \\ &= \mu\frac{\partial}{\partial C}(III_C^{-1/3}C : I) + pJC^{-1} \\ &= \mu[III_C^{-1/3}I - \tfrac{1}{3}III_C^{-1/3-1}III_C C^{-1}(C : I)] + pJC^{-1} \\ &= \mu III_C^{-1/3}(I - \tfrac{1}{3}I_C C^{-1}) + pJC^{-1} \end{aligned} \tag{5.50}$$

The corresponding Cauchy stress tensor can now be obtained by using Equation (4.45b) to give $\sigma$ as,

$$\begin{aligned} \sigma &= J^{-1}FSF^T \\ &= \mu J^{-5/3}F(I - \tfrac{1}{3}I_C C^{-1})F^T + pFC^{-1}F^T \\ &= \sigma' + pI\,; \qquad \sigma' = \mu J^{-5/3}(b - \tfrac{1}{3}I_b I) \end{aligned} \tag{5.51}$$

where the fact that $I_b = I_C$ has been used again.

We can now evaluate the Lagrangian elasticity tensor with the help of Equations (5.10) or (5.11). The result can be split into deviatoric and pressure components, $\widehat{\mathcal{C}}$ and $\mathcal{C}_p$ respectively, as,

$$\mathcal{C} = 2\frac{\partial S}{\partial C} = \widehat{\mathcal{C}} + \mathcal{C}_p; \qquad \widehat{\mathcal{C}} = 2\frac{\partial S'}{\partial C} = 4\frac{\partial^2 \widehat{\Psi}}{\partial C \partial C}; \qquad \mathcal{C}_p = 2p\frac{\partial (JC^{-1})}{\partial C} \tag{5.52}$$

With the help of Equations (5.22) and (5.31) these two components can be evaluated for the neo-Hookean case defined by Equation (5.49) after lengthy but simple algebra as,

$$\widehat{\mathcal{C}} = 2\mu III_C^{-1/3}\left[\tfrac{1}{3}I_C \mathcal{I} - \tfrac{1}{3}I \otimes C^{-1} - \tfrac{1}{3}C^{-1} \otimes I + \tfrac{1}{9}I_C C^{-1} \otimes C^{-1}\right] \tag{5.53a}$$

$$\mathcal{C}_p = pJ[C^{-1} \otimes C^{-1} - 2\mathcal{I}] \tag{5.53b}$$

Note that the pressure component $\mathcal{C}_p$ does not depend on the particular material definition being used.

The spatial elasticity tensor is obtained by the push forward type of operation shown in Equation (5.14) as,

$$\mathcal{c} = \hat{\mathcal{c}} + \mathcal{c}_p; \qquad \hat{\mathcal{c}} = J^{-1}\phi_*[\widehat{\mathcal{C}}]; \qquad \mathcal{c}_p = J^{-1}\phi_*[\mathcal{C}_p] \tag{5.54}$$

Performing this push forward operation in Equations (5.53a,b) gives,

$$\hat{\mathcal{c}} = 2\mu J^{-5/3}\left[\tfrac{1}{3}I_C \boldsymbol{i} - \tfrac{1}{3}b \otimes I - \tfrac{1}{3}I \otimes b + \tfrac{1}{9}I_b I \otimes I\right] \tag{5.55a}$$

$$\mathcal{c}_p = p[I \otimes I - 2\boldsymbol{i}] \tag{5.55b}$$

**EXAMPLE 5.5: Mooney–Rivlin materials**

A general form for the strain energy function of incompressible rubbers attributable to Mooney and Rivlin is expressed as,

$$\Psi(C) = \sum_{r,s\geq 0} \mu_{rs}(I_C - 3)^r (II_C^* - 3)^s$$

where $II_C^*$ is the second invariant of $C$ defined as,

$$II_C^* = \tfrac{1}{2}(I_C^2 - II_C); \qquad II_C = C : C$$

The most frequently used of this family of equations is obtained when only $\mu_{01}$ and $\mu_{10}$ are different from zero. In this particular case we have,

$$\Psi(C) = \mu_{10}(I_C - 3) + \tfrac{1}{2}\mu_{01}(I_C^2 - II_C - 6)$$

*(continued)*



**EXAMPLE 5.5** *(cont.)*

The equivalent homogeneous potential is obtained by replacing $C$ by $\hat{C}$ in this equation to give,

$$\widehat{\Psi}(C) = \mu_{10}(\mathrm{tr}\,\hat{C} - 3) + \tfrac{1}{2}\mu_{01}[(\mathrm{tr}\,\hat{C})^2 - \hat{C} : \hat{C} - 6]$$

### 5.5.3 Nearly Incompressible Hyperelastic Materials

As explained at the beginning of Section 5.5 near incompressibility is often a device by which incompressibility can more readily be enforced within the context of the finite element formulation. This is facilitated by adding a volumetric energy component $U(J)$ to the distortional component $\widehat{\Psi}$ already defined to give the total strain energy function $\Psi(C)$ as,

$$\Psi(C) = \widehat{\Psi}(C) + U(J) \tag{5.56}$$

where the simplest example of a volumetric function $U(J)$ is,

$$U(J) = \tfrac{1}{2}\kappa(J-1)^2 \tag{5.57}$$

It will be seen in Chapter 6 that when equilibrium is expressed in a variational framework, the use of Equation (5.57) with a large so-called penalty number $\kappa$ will approximately enforce incompressibility. Typically, values of $\kappa$ in the region of $10^3 - 10^4\,\mu$ are used for this purpose. Nevertheless, we must emphasize that $\kappa$ can represent a true material property, namely the bulk modulus, for a compressible material that happens to have a hyperelastic strain energy function in the form given by Equations (5.56) and (5.57).

The second Piola–Kirchhoff tensor for a material defined by (5.56) is obtained in the standard manner with the help of Equation (5.22) and noting that $III_C = J^2$ to give,

$$\begin{aligned} S &= 2\frac{\partial \Psi}{\partial C} \\ &= 2\frac{\partial \widehat{\Psi}}{\partial C} + 2\frac{dU}{dJ}\frac{\partial J}{\partial C} \\ &= 2\frac{\partial \widehat{\Psi}}{\partial C} + pJC^{-1} \end{aligned} \tag{5.58}$$

where, by comparison with (5.46), we have identified the pressure as,

$$p = \frac{dU}{dJ} \tag{5.59}$$

which for the case where $U(J)$ is given by Equation (5.57) gives,

$$p = k(J-1) \tag{5.60}$$

This value of the pressure can be substituted into the general Equation (5.58) or into the particular Equation (5.50) for the neo-Hookean case to yield the complete second Piola–Kirchhoff tensor. Alternatively, in the neo-Hookean case, $p$ can be substituted into Equation (5.51) to give the Cauchy stress tensor.

### EXAMPLE 5.6: Simple shear (ii)

Again we can study the case of simple shear for a nearly incompressible neo-Hookean material. Using Equation (5.51) and the $\boldsymbol{b}$ tensor given in Exercise 5.4 we obtain,

$$\boldsymbol{\sigma} = \mu \begin{bmatrix} \frac{2}{3}\gamma^2 & \gamma & 0 \\ \gamma & -\frac{1}{3}\gamma^2 & 0 \\ 0 & 0 & -\frac{1}{3}\gamma^2 \end{bmatrix}$$

where now the pressure is zero as $J = 1$ for this type of deformation. Note that for this type of material there is no Kelvin effect in the sense that a volume-preserving motion leads to a purely deviatoric stress tensor.

### EXAMPLE 5.7: Pure dilatation (ii)

It is also useful to examine the consequences of a pure dilatation on a nearly incompressible material. Recalling that this type of deformation has an associated left Cauchy–Green tensor $\boldsymbol{b} = J^{2/3}\boldsymbol{I}$ whose trace is $I_b = 3J^{2/3}$, Equations (5.51) and (5.60) give,

$$\boldsymbol{\sigma} = \kappa(J-1)\boldsymbol{I}$$

As expected a purely dilatational deformation leads to a hydrostatic state of stresses. Note also that the isochoric potential $\widehat{\Psi}$ plays no role in the value of the pressure $p$.

Again, to complete the description of this type of material it is necessary to derive the Lagrangian and spatial elasticity tensors. The Lagrangian tensor can be split into three components given as,

$$\boldsymbol{\mathcal{C}} = 4\frac{\partial \widehat{\Psi}}{\partial \boldsymbol{C}\partial \boldsymbol{C}} + 2p\frac{\partial (J\boldsymbol{C}^{-1})}{\partial \boldsymbol{C}} + 2J\boldsymbol{C}^{-1} \otimes \frac{\partial p}{\partial \boldsymbol{C}} \tag{5.61}$$

The first two components in this expression are $\widehat{\boldsymbol{\mathcal{C}}}$ and $\boldsymbol{\mathcal{C}}_p$ as evaluated in the previous section in Equations (5.53a,b). The final term, namely $\boldsymbol{\mathcal{C}}_\kappa$, represents a volumetric



tangent component and follows from $U(J)$ and Equation (5.22) as,

$$\begin{aligned}\boldsymbol{\mathcal{C}}_\kappa &= 2J\boldsymbol{C}^{-1} \otimes \frac{\partial p}{\partial \boldsymbol{C}} \\ &= 2J\frac{d^2U}{dJ^2}\boldsymbol{C}^{-1} \otimes \frac{\partial J}{\partial \boldsymbol{C}} \\ &= J^2\frac{d^2U}{dJ^2}\boldsymbol{C}^{-1} \otimes \boldsymbol{C}^{-1}\end{aligned} \tag{5.62}$$

which in the case $U(J) = \kappa(J-1)^2/2$ becomes,

$$\boldsymbol{\mathcal{C}}_\kappa = \kappa J^2 \boldsymbol{C}^{-1} \otimes \boldsymbol{C}^{-1} \tag{5.63}$$

Finally, the spatial elasticity tensor is obtained by standard push forward operation to yield,

$$\boldsymbol{c} = J^{-1}\phi_*[\boldsymbol{\mathcal{C}}] = \hat{\boldsymbol{c}} + \boldsymbol{c}_p + \boldsymbol{c}_\kappa \tag{5.64}$$

where the deviatoric and pressure components, $\hat{\boldsymbol{c}}$ and $\boldsymbol{c}_p$ respectively, are identical to those derived in the previous section and the volumetric component $\boldsymbol{c}_\kappa$ is,

$$\boldsymbol{c}_\kappa = J^{-1}\phi_*[\boldsymbol{\mathcal{C}}_\kappa] = J\frac{d^2U}{dJ^2}\boldsymbol{I} \otimes \boldsymbol{I} \tag{5.65}$$

which for the particular function $U(J)$ defined in Equation (5.57) gives,

$$\boldsymbol{c}_\kappa = \kappa J\, \boldsymbol{I} \otimes \boldsymbol{I} \tag{5.66}$$

*Remark 3:* At the initial configuration, $\boldsymbol{F} = \boldsymbol{C} = \boldsymbol{b} = \boldsymbol{I}$, $J = 1$, $p = 0$, and the above elasticity tensor becomes,

$$\begin{aligned}\boldsymbol{c} &= \hat{\boldsymbol{c}} + \boldsymbol{c}_\kappa \\ &= 2\mu\left[\boldsymbol{\iota} - \tfrac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right] + \kappa \boldsymbol{I}\otimes\boldsymbol{I} \\ &= \left(\kappa - \tfrac{2}{3}\mu\right)\boldsymbol{I}\otimes\boldsymbol{I} + 2\mu\boldsymbol{\iota}\end{aligned} \tag{5.67}$$

which coincides with the standard spatially isotropic elasticity tensor (5.37) with the relationship between $\lambda$ and $\kappa$ given as,

$$\lambda = \kappa - \tfrac{2}{3}\mu \tag{5.68}$$

In fact all isotropic hyperelastic materials have initial elasticity tensors as defined by Equation (5.37).

## 5.6 ISOTROPIC ELASTICITY IN PRINCIPAL DIRECTIONS

### 5.6.1 Material Description

It is often the case that the constitutive equations of a material are presented in terms of the stretches $\lambda_1$, $\lambda_2$, $\lambda_3$ in the principal directions $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, and $\boldsymbol{N}_3$ as defined in Section 3.5. In the case of hyperelasticity, this assumes that the stored elastic energy function is obtainable in terms of $\lambda_\alpha$ rather than the invariants of $\boldsymbol{C}$. This is most likely to be the case in the experimental determination of the constitutive parameters.

In order to obtain the second Piola–Kirchhoff stress in terms of the principal directions and stretches, recall Equation (5.23) and note that the identity, the right Cauchy–Green tensor, and its inverse can be expresses as (see Equations (2.30) and (3.30)),

$$\boldsymbol{I} = \sum_{\alpha=1}^{3} \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \tag{5.69a}$$

$$\boldsymbol{C} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \tag{5.69b}$$

$$\boldsymbol{C}^{-1} = \sum_{\alpha=1}^{3} \lambda_\alpha^{-2} \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \tag{5.69c}$$

Substituting these equations into Equation (5.23) gives $\boldsymbol{S}$ as,

$$\boldsymbol{S} = \sum_{I=1}^{3} (2\Psi_I + 4\Psi_{II}\lambda_\alpha^2 + 2III_C\Psi_{III}\lambda_\alpha^{-2})\boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \tag{5.70}$$

Given that the term in brackets is a scalar it is immediately apparent that for an isotropic material the principal axes of stress coincide with the principal axes of strain. The terms $\Psi_I$, $\Psi_{II}$, and $\Psi_{III}$ in Equation (5.70) refer to the derivatives with respect to the invariants of $\boldsymbol{C}$. Hence it is necessary to transform these into derivatives with respect to the stretches. For this purpose note that the squared stretches $\lambda_\alpha^2$ are the eigenvalues of $\boldsymbol{C}$, which according to the general relationships (2.60a–c) are related to the invariants of $\boldsymbol{C}$ as,

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{5.71a}$$



$$II_C = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 \tag{5.71b}$$

$$III_C = \lambda_1^2 \lambda_2^2 \lambda_3^2 \tag{5.71c}$$

Differentiating these equations gives,

$$1 = \frac{\partial I_C}{\partial \lambda_\alpha^2} \tag{5.72a}$$

$$2\lambda_\alpha^2 = \frac{\partial II_C}{\partial \lambda_\alpha^2} \tag{5.72b}$$

$$\frac{III_C}{\lambda_\alpha^2} = \frac{\partial III_C}{\partial \lambda_\alpha^2} \tag{5.72c}$$

which upon substitution into Equation (5.70) and using the chain rule gives the principal components of the second Piola–Kirchhoff tensor as derivatives of $\Psi$ with respect to the principal stretches as,

$$\boldsymbol{S} = \sum_{\alpha=1}^{3} S_{\alpha\alpha} \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha; \qquad S_{\alpha\alpha} = 2\frac{\partial \Psi}{\partial \lambda_\alpha^2} \tag{5.73}$$

### 5.6.2 Spatial Description

In order to obtain an equation analogous to (5.73) for the Cauchy stress, substitute this equation into Equation (4.45b) to give,

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{FSF}^T = \sum_{\alpha=1}^{3} \frac{2}{J}\frac{\partial \Psi}{\partial \lambda_\alpha^2}(\boldsymbol{FN}_\alpha)\otimes(\boldsymbol{FN}_\alpha) \tag{5.74}$$

Observing from Equation (3.44a) that $\boldsymbol{FN}_\alpha = \lambda_\alpha \boldsymbol{n}_\alpha$ yields the principal components of Cauchy stress tensor after simple algebra as,

$$\boldsymbol{\sigma} = \sum_{\alpha=1}^{3} \sigma_{\alpha\alpha} \boldsymbol{n}_\alpha \otimes \boldsymbol{n}_\alpha; \qquad \sigma_{\alpha\alpha} = \frac{\lambda_\alpha}{J}\frac{\partial \Psi}{\partial \lambda_\alpha} = \frac{1}{J}\frac{\partial \Psi}{\partial \ln \lambda_\alpha} \tag{5.75}$$

The evaluation of the Cartesian components of the Cauchy stress can be easily achieved by interpreting Equation (5.75) in a matrix form using Equation (2.40d) for the components of the tensor product to give,

$$[\boldsymbol{\sigma}] = \sum_{\alpha=1}^{3} \sigma_{\alpha\alpha}[\boldsymbol{n}_\alpha][\boldsymbol{n}_\alpha]^T \tag{5.76}$$

where $[\boldsymbol{\sigma}]$ denotes the matrix formed by the Cartesian components of $\boldsymbol{\sigma}$ and $[\boldsymbol{n}_\alpha]$

are the column vectors containing the Cartesian components of $\boldsymbol{n}_\alpha$. Alternatively, a similar evaluation can be performed in an indicial manner by introducing $T_{\alpha j}$ as the Cartesian components of $\boldsymbol{n}_\alpha$, that is, $\boldsymbol{n}_\alpha = \sum_{j=1}^{3} T_{\alpha j}\boldsymbol{e}_j$, and substituting into Equation (5.75) to give,

$$\boldsymbol{\sigma} = \sum_{j,k=1}^{3}\left(\sum_{\alpha=1}^{3}\sigma_{\alpha\alpha}T_{\alpha j}T_{\alpha k}\right)\boldsymbol{e}_j\otimes\boldsymbol{e}_k \tag{5.77}$$

The expression in brackets in the above equation gives again the Cartesian components of the Cauchy stress tensor.

### 5.6.3 Material Elasticity Tensor

To construct the material elasticity tensor for a material given in terms of the principal stretches it is again temporarily convenient to consider the time derivative Equation (5.12), that is, $\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{C}} : \dot{\boldsymbol{E}}$. From Equation (3.123) it transpires that $\dot{\boldsymbol{E}}$ can be written in principal directions as,

$$\dot{\boldsymbol{E}} = \sum_{\alpha=1}^{3}\frac{1}{2}\frac{d\lambda_\alpha^2}{dt}\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\alpha + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}\frac{1}{2}W_{\alpha\beta}(\lambda_\alpha^2-\lambda_\beta^2)\,\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta \tag{5.78}$$

where $W_{\alpha\beta}$ are the components of the spin tensor of the Lagrangian triad, that is, $\dot{\boldsymbol{N}}_\alpha = \sum_{\beta=1}^{3} W_{\alpha\beta}\boldsymbol{N}_\beta$. A similar expression for the time derivative of $\boldsymbol{S}$ can be obtained by differentiating Equation (5.73) to give,

$$\begin{aligned}\dot{\boldsymbol{S}} &= \sum_{\alpha,\beta=1}^{3}2\frac{\partial^2\Psi}{\partial\lambda_\alpha^2\partial\lambda_\beta^2}\frac{d\lambda_\beta^2}{dt}\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\alpha + \sum_{\alpha=1}^{3}2\frac{\partial\Psi}{\partial\lambda_\alpha^2}(\dot{\boldsymbol{N}}_\alpha\otimes\boldsymbol{N}_\alpha+\boldsymbol{N}_\alpha\otimes\dot{\boldsymbol{N}}_\alpha)\\ &= \sum_{\alpha,\beta=1}^{3}2\frac{\partial^2\Psi}{\partial\lambda_\alpha^2\partial\lambda_\beta^2}\frac{d\lambda_\beta^2}{dt}\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\alpha + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}(S_{\alpha\alpha}-S_{\beta\beta})W_{\alpha\beta}\,\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\end{aligned} \tag{5.79}$$

Now observe from Equation (5.78) that the on-diagonal and off-diagonal terms of $\dot{\boldsymbol{E}}$ are,

$$\frac{d\lambda_\alpha^2}{dt} = 2\dot{E}_{\alpha\alpha} \tag{5.80a}$$

$$W_{\alpha\beta} = \frac{2\dot{E}_{\alpha\beta}}{\lambda_\alpha^2-\lambda_\beta^2};\quad (\alpha\neq\beta) \tag{5.80b}$$

Substituting Equations (5.80a–b) into (5.79) and expressing the components of $\dot{\boldsymbol{E}}$ as $\dot{E}_{\alpha\beta} = (\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta) : \dot{\boldsymbol{E}}$ yields,



$$\begin{aligned}\dot{\boldsymbol{S}} &= \sum_{\alpha=1}^{3}4\frac{\partial^2\Psi}{\partial\lambda_\alpha^2\partial\lambda_\beta^2}\dot{E}_{\beta\beta}(\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\alpha) + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}2\frac{S_{\alpha\alpha}-S_{\beta\beta}}{\lambda_\alpha^2-\lambda_\beta^2}\dot{E}_{\alpha\beta}\,\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\\ &= \left[\sum_{\alpha,\beta=1}^{3}\frac{4\partial^2\Psi}{\partial\lambda_\alpha^2\partial\lambda_\beta^2}\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\otimes\boldsymbol{N}_\beta\right.\\ &\quad\left.+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}2\frac{S_{\alpha\alpha}-S_{\beta\beta}}{\lambda_\alpha^2-\lambda_\beta^2}\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\otimes\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\right]:\dot{\boldsymbol{E}}\end{aligned} \tag{5.81}$$

Comparing this expression with the rate equation $\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{C}} : \dot{\boldsymbol{E}}$, the material or Lagrangian elasticity tensor emerges as,

$$\begin{aligned}\boldsymbol{\mathcal{C}} &= \sum_{\alpha,\beta=1}^{3}4\frac{\partial^2\Psi}{\partial\lambda_\alpha^2\partial\lambda_\beta^2}\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\otimes\boldsymbol{N}_\beta\\ &\quad+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}2\,\frac{S_{\alpha\alpha}-S_{\beta\beta}}{\lambda_\alpha^2-\lambda_\beta^2}\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\otimes\boldsymbol{N}_\alpha\otimes\boldsymbol{N}_\beta\end{aligned} \tag{5.82}$$

*Remark 4:* In the particular case when $\lambda_\alpha = \lambda_\beta$ isotropy implies that $S_{\alpha\alpha} = S_{\beta\beta}$, and the quotient $(S_{\alpha\alpha}-S_{\beta\beta})/(\lambda_\alpha^2-\lambda_\beta^2)$ in Equation (5.82) must be evaluated using L'Hospital's rule to give,

$$\lim_{\lambda_\beta\to\lambda_\alpha}2\,\frac{S_{\alpha\alpha}-S_{\beta\beta}}{\lambda_\alpha^2-\lambda_\beta^2} = 4\left[\frac{\partial^2\Psi}{\partial\lambda_\beta^2\partial\lambda_\beta^2}-\frac{\partial^2\Psi}{\partial\lambda_\alpha^2\partial\lambda_\beta^2}\right] \tag{5.83}$$

### 5.6.4 Spatial Elasticity Tensor

The spatial elasticity tensor is obtained by pushing the Lagrangian tensor forward to the current configuration using Equation (5.14), which involves the product by $\boldsymbol{F}$ four times as,

$$\begin{aligned}\boldsymbol{c} &= \sum_{\alpha,\beta=1}^{3}\frac{1}{J}\frac{\partial^2\Psi}{\partial\lambda_\alpha^2\partial\lambda_\beta^2}(\boldsymbol{F}\boldsymbol{N}_\alpha)\otimes(\boldsymbol{F}\boldsymbol{N}_\alpha)\otimes(\boldsymbol{F}\boldsymbol{N}_\beta)\otimes(\boldsymbol{F}\boldsymbol{N}_\beta)\\ &\quad+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3}\frac{2}{J}\frac{S_{\alpha\alpha}-S_{\beta\beta}}{\lambda_\alpha^2-\lambda_\beta^2}(\boldsymbol{F}\boldsymbol{N}_\alpha)\otimes(\boldsymbol{F}\boldsymbol{N}_\beta)\otimes(\boldsymbol{F}\boldsymbol{N}_\alpha)\otimes(\boldsymbol{F}\boldsymbol{N}_\beta)\end{aligned} \tag{5.84}$$

Noting again that $FN_\alpha = \lambda_\alpha n_\alpha$ and after some algebraic manipulations using the standard chain rule we can eventually derive the Eulerian or spatial elasticity tensor as,

$$\mathcal{c} = \sum_{\alpha,\beta=1}^{3} \frac{1}{J}\frac{\partial^2 \Psi}{\partial \ln\lambda_\alpha \partial \ln\lambda_\beta} n_\alpha \otimes n_\alpha \otimes n_\beta \otimes n_\beta - \sum_{\alpha=1}^{3} 2\sigma_{\alpha\alpha} n_\alpha \otimes n_\alpha \otimes n_\alpha \otimes n_\alpha$$

$$+ \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3} 2\frac{\sigma_{\alpha\alpha}\lambda_\beta^2 - \sigma_{\beta\beta}\lambda_\alpha^2}{\lambda_\alpha^2 - \lambda_\beta^2} n_\alpha \otimes n_\beta \otimes n_\alpha \otimes n_\beta \tag{5.85}$$

The evaluation of the Cartesian components of this tensor requires a similar transformation to that employed in Equation (5.77) for the Cauchy stresses. Using the same notation, the Cartesian components of the Eulerian triad $T_{\alpha j}$ are substituted into Equation (5.85) to give after simple algebra the Cartesian components of $\mathcal{c}$ as,

$$\mathcal{c}_{ijkl} = \sum_{\alpha,\beta=1}^{3} \frac{1}{J}\frac{\partial^2 \Psi}{\partial \ln\lambda_\alpha \partial \ln\lambda_\beta} T_{\alpha i}T_{\alpha j}T_{\beta k}T_{\beta l} - \sum_{\alpha=1}^{3} 2\sigma_{\alpha\alpha} T_{\alpha i}T_{\alpha j}T_{\alpha k}T_{\alpha l}$$

$$+ \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{3} 2\frac{\sigma_{\alpha\alpha}\lambda_\beta^2 - \sigma_{\beta\beta}\lambda_\alpha^2}{\lambda_\alpha^2 - \lambda_\beta^2} T_{\alpha i}T_{\beta j}T_{\alpha k}T_{\beta l} \tag{5.86}$$

*Remark 5:* Again, recalling Remark 4, in the case when $\lambda_\alpha = \lambda_\beta$, L'Hospital's rule yields,

$$\lim_{\lambda_\beta \to \lambda_\alpha} 2\frac{\sigma_{\alpha\alpha}\lambda_\beta^2 - \sigma_{\beta\beta}\lambda_\alpha^2}{\lambda_\alpha^2 - \lambda_\beta^2} = \frac{1}{J}\left[\frac{\partial^2\Psi}{\partial \ln\lambda_\beta \partial \ln\lambda_\beta} - \frac{\partial^2\Psi}{\partial \ln\lambda_\alpha \partial \ln\lambda_\beta}\right] - 2\sigma_{\beta\beta} \tag{5.87}$$

## 5.6.5 A Simple Stretch-Based Hyperelastic Material

A material frequently encountered in the literature is defined by a hyperelastic potential in terms of the logarithmic stretches and two material parameters $\lambda$ and $\mu$ as,

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \mu[(\ln\lambda_1)^2 + (\ln\lambda_2)^2 + (\ln\lambda_3)^2] + \frac{\lambda}{2}(\ln J)^2 \tag{5.88}$$

where, because $J = \lambda_1\lambda_2\lambda_3$,

$$\ln J = \ln\lambda_1 + \ln\lambda_2 + \ln\lambda_3 \tag{5.89}$$

It will be shown that the potential $\Psi$ leads to a generalization of the stress–strain relationships employed in classical linear elasticity.



Using Equation (5.75) the principal Cauchy stress components emerge as,

$$\sigma_{\alpha\alpha} = \frac{1}{J}\frac{\partial\Psi}{\partial \ln\lambda_\alpha} = \frac{2\mu}{J}\ln\lambda_\alpha + \frac{\lambda}{J}\ln J \tag{5.90}$$

Furthermore, the coefficients of the elasticity tensor in (5.86) are,

$$\frac{1}{J}\frac{\partial^2\Psi}{\partial \ln\lambda_\alpha \partial \ln\lambda_\beta} = \frac{\lambda}{J} + \frac{2\mu}{J}\delta_{\alpha\beta} \tag{5.91}$$

The similarities between these equations and linear elasticity can be established if we first recall the standard small strain elastic equations as,

$$\sigma_{\alpha\alpha} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{\alpha\alpha} \tag{5.92}$$

Recalling that $\ln J = \ln\lambda_1 + \ln\lambda_2 + \ln\lambda_3$ it transpires that Equations (5.90) and (5.92) are identical except for the small strains having been replaced by the logarithmic stretches and $\lambda$ and $\mu$ by $\lambda/J$ and $\mu/J$ respectively. The stress–strain equations can be inverted and expressed in terms of the more familiar material parameters $E$ and $\nu$, the Young's modulus and Poisson ratio, as,

$$\ln\lambda_\alpha = \frac{J}{E}[(1+\nu)\sigma_{\alpha\alpha} - \nu(\sigma_{11} + \sigma_{22} + \sigma_{33})]; \qquad E = \frac{\mu(2\mu + 3\lambda)}{\lambda + \mu};$$

$$\nu = \frac{\lambda}{2\lambda + 2\mu} \tag{5.93a,b,c}$$

*Remark 6:* At the initial unstressed configuration, $J = \lambda_\alpha = 1$, $\sigma_{\alpha\alpha} = 0$, and the principal directions coincide with the three spatial directions $n_\alpha = e_\alpha$ and therefore $T_{\alpha j} = \delta_{\alpha j}$. Substituting these values into Equations (5.91), (5.87), and (5.86) gives the initial elasticity tensor for this type of material as,

$$\mathcal{c}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl} \tag{5.94}$$

which again (see Remark 3) coincides with the standard spatially isotropic elasticity tensor.

## 5.6.6 Nearly Incompressible Material in Principal Directions

In view of the importance of nearly incompressible material behavior, coupled with the likelihood that such materials will be described naturally in terms of principal stretches, it is now logical to elaborate the formulation in preparation for the case when the material defined by Equation (5.88) becomes nearly incompressible. Once again, the distortional components of the kinematic variables being used, namely the stretches $\lambda_\alpha$, must be identified first. This is achieved by recalling Equations (3.43) and (3.61) for $F$ and $\hat{F}$ to give,

$$\begin{aligned}\hat{F} &= J^{-1/3}F \\ &= J^{-1/3}\sum_{\alpha=1}^{3}\lambda_\alpha\, n_\alpha \otimes N_\alpha \\ &= \sum_{\alpha=1}^{3}(J^{-1/3}\lambda_\alpha)\, n_\alpha \otimes N_\alpha\end{aligned} \tag{5.95}$$

This enables the distortional stretches $\hat{\lambda}_\alpha$ to be identified as,

$$\hat{\lambda}_\alpha = J^{-1/3}\lambda_\alpha; \qquad \lambda_\alpha = J^{1/3}\hat{\lambda}_\alpha \tag{5.96a,b}$$

Substituting (5.96b) into the hyperelastic potential defined in (5.88) yields after simple algebra a decoupled representation of this material as,

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \widehat{\Psi}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) + U(J) \tag{5.97}$$

where the distortional and volumetric components are,

$$\widehat{\Psi}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) = \mu[(\ln\hat{\lambda}_1)^2 + (\ln\hat{\lambda}_2)^2 + (\ln\hat{\lambda}_3)^2] \tag{5.98a}$$

$$U(J) = \tfrac{1}{2}\kappa(\ln J)^2; \qquad \kappa = \lambda + \tfrac{2}{3}\mu \tag{5.98b}$$

Note that this equation is a particular case of the decoupled Equation (5.56) with alternative definitions of $U(J)$ and $\widehat{\Psi}$. The function $U(J)$ will enforce incompressibility only when the ratio $\kappa$ to $\mu$ is sufficiently high, typically $10^3$–$10^4$. Under such conditions the value of $J$ is $J \approx 1$ and $\ln J \approx 1 - J$, and therefore the value of $U$ will approximately coincide with the function defined in (5.57).

For the expression $U(J)$, the corresponding value of the hydrostatic pressure $p$ is re-evaluated using Equation (5.59) to give,

$$p = \frac{dU}{dJ} = \frac{\kappa \ln J}{J} \tag{5.99}$$

In order to complete the stress description, the additional deviatoric component must be evaluated by recalling Equation (5.75) as,

$$\begin{aligned}\sigma_{\alpha\alpha} &= \frac{1}{J}\frac{\partial\Psi}{\partial\ln\lambda_\alpha} \\ &= \frac{1}{J}\frac{\partial\widehat{\Psi}}{\partial\ln\lambda_\alpha} + \frac{1}{J}\frac{\partial U}{\partial\ln\lambda_\alpha} \\ &= \frac{1}{J}\frac{\partial\widehat{\Psi}}{\partial\ln\lambda_\alpha} + \frac{\kappa\ln J}{J}\end{aligned} \tag{5.100}$$



Observing that the second term in this equation is the pressure, the principal deviatoric stress components are obviously,

$$\sigma'_{\alpha\alpha} = \frac{1}{J}\frac{\partial\widehat{\Psi}}{\partial\ln\lambda_\alpha} \tag{5.101}$$

In order to obtain the derivatives of $\widehat{\Psi}$ it is convenient to rewrite this function with the help of Equation (5.96a) as,

$$\begin{aligned}\widehat{\Psi} &= \mu[(\ln\hat{\lambda}_1)^2 + (\ln\hat{\lambda}_2)^2 + (\ln\hat{\lambda}_3)^2] \\ &= \mu[(\ln\lambda_1)^2 + (\ln\lambda_2)^2 + (\ln\lambda_3)^2] + \tfrac{1}{3}\mu(\ln J)^2 \\ &\quad - \tfrac{2}{3}\mu(\ln J)(\ln\lambda_1 + \ln\lambda_2 + \ln\lambda_3) \\ &= \mu[(\ln\lambda_1)^2 + (\ln\lambda_2)^2 + (\ln\lambda_3)^2] - \tfrac{1}{3}\mu(\ln J)^2\end{aligned} \tag{5.102}$$

This expression for $\widehat{\Psi}$ is formally identical to Equation (5.88) for the complete hyperelastic potential $\Psi$ with the Lamè coefficient $\lambda$ now replaced by $-2\mu/3$. Consequently, Equation (5.90) can now be recycled to give the deviatoric principal stress component as,

$$\sigma'_{\alpha\alpha} = \frac{2\mu}{J}\ln\lambda_\alpha - \frac{2\mu}{3J}\ln J \tag{5.103}$$

The final stage in this development is the evaluation of the volumetric and deviatoric components of the spatial elasticity tensor $\mathcal{c}$. For a general decoupled hyperelastic potential this decomposition is embodied in Equation (5.64), where $\mathcal{c}$ is expressed as,

$$\mathcal{c} = \hat{\mathcal{c}} + \mathcal{c}_p + \mathcal{c}_\kappa \tag{5.104}$$

where the origin of the pressure component $\mathcal{c}_p$ is the second term in the general equation for the Lagrangian elasticity tensor (5.61), which is entirely geometrical, that is, independent of the material being used, and therefore remains unchanged as given by Equation (5.55b). However, the volumetric component $\mathcal{c}_\kappa$ depends on the particular function $U(J)$ being used and in the present case becomes,

$$\begin{aligned}\mathcal{c}_\kappa &= J\frac{d^2U}{dJ^2}\, I \otimes I \\ &= \frac{\kappa(1 - pJ)}{J}\, I \otimes I\end{aligned} \tag{5.105}$$

The deviatoric component of the elasticity tensor $\hat{\mathcal{c}}$ emerges from the push forward of the first term in Equation (5.61). Its evaluation is facilitated by again recalling that $\widehat{\Psi}$ coincides with $\Psi$ when the parameter $\lambda$ is replaced by $-2\mu/3$. A reformulation of the spatial elasticity tensor following the procedure previously described with this substitution and the corresponding replacement of $\sigma_{\alpha\alpha}$ by $\sigma'_{\alpha\alpha}$ inevitably leads to the Cartesian components of $\mathfrak{c}$ as,

$$\mathfrak{c}_{ijkl} = \sum_{\alpha,\beta=1}^{3} \frac{1}{J} \frac{\partial^2 \widehat{\Psi}}{\partial \ln \lambda_\alpha \partial \ln \lambda_\beta} T_{\alpha i} T_{\alpha j} T_{\beta k} T_{\beta l} - \sum_{\alpha=1}^{3} 2\sigma'_{\alpha\alpha} T_{\alpha i} T_{\alpha j} T_{\alpha k} T_{\alpha l}$$

$$+ \sum_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{3} 2 \frac{\sigma'_{\alpha\alpha} \lambda_\beta^2 - \sigma'_{\beta\beta} \lambda_\alpha^2}{\lambda_\alpha^2 - \lambda_\beta^2} T_{\alpha i} T_{\beta j} T_{\alpha k} T_{\beta l} \tag{5.106}$$

where the derivatives of $\widehat{\Psi}$ for the material under consideration are,

$$\frac{1}{J} \frac{\partial^2 \widehat{\Psi}}{\partial \ln \lambda_\alpha \partial \ln \lambda_\beta} = \frac{2\mu}{J} \delta_{\alpha\beta} - \frac{2\mu}{3J} \tag{5.107}$$

### 5.6.7 Plane Strain and Plane Stress Cases

The plane strain case is defined by the fact that the stretch in the third direction $\lambda_3 = 1$. Under such conditions, the stored elastic potential becomes,

$$\Psi(\lambda_1, \lambda_2) = \mu[(\ln \lambda_1)^2 + (\ln \lambda_2)^2] + \frac{\lambda}{2} (\ln j)^2 \tag{5.108}$$

where $j = \det_{2\times 2} \boldsymbol{F}$ is the determinant of the components of $\boldsymbol{F}$ in the $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ plane. The three stresses are obtained using exactly Equation (5.90) with $\lambda_3 = 1$ and $J = j$.

The plane stress case is a little more complicated in that it is the stress in the $\boldsymbol{n}_3$ direction rather than the stretch that is constrained, that is $\sigma_{33} = 0$. Imposing this condition in Equation (5.90) gives,

$$\sigma_{33} = 0 = \frac{\lambda}{J} \ln J + \frac{2\mu}{J} \ln \lambda_3 \tag{5.109}$$

from which the logarithmic stretch in the third direction emerges as,

$$\ln \lambda_3 = -\frac{\lambda}{\lambda + 2\mu} \ln j \tag{5.110}$$

Substituting this expression into Equation (5.88) and noting that $\ln J = \ln \lambda_3 + \ln j$ gives,

$$\Psi(\lambda_1, \lambda_2) = \mu[(\ln \lambda_1)^2 + (\ln \lambda_2)^2] + \frac{\bar{\lambda}}{2} (\ln j)^2 \tag{5.111}$$

where the effective Lame coefficient $\bar{\lambda}$ is,

$$\bar{\lambda} = \gamma \lambda; \qquad \gamma = \frac{2\mu}{\lambda + 2\mu} \tag{5.112}$$



Additionally, using Equation (5.110) the three-dimensional volume ratio $J$ can be found as a function of the planar component $j$ as,

$$J = j^\gamma \tag{5.113}$$

By either substituting Equation (5.110) into Equation (5.88) or differentiating Equation (5.111) the principal Cauchy stress components are obtained as,

$$\sigma_{\alpha\alpha} = \frac{\bar{\lambda}}{j^\gamma} \ln j + \frac{2\mu}{j^\gamma} \ln \lambda_\alpha \tag{5.114}$$

and the coefficients of the Eulerian elasticity tensor become,

$$\frac{1}{J} \frac{\partial^2 \Psi}{\partial \ln \lambda_\alpha \partial \ln \lambda_\beta} = \frac{\bar{\lambda}}{j^\gamma} + \frac{2\mu}{j^\gamma} \delta_{\alpha\beta} \tag{5.115}$$

### 5.6.8 Uniaxial Rod Case

In a uniaxial rod case, the stresses in directions orthogonal to the rod, $\sigma_{22}$ and $\sigma_{33}$ vanish. Imposing this condition in Equation (5.90) gives two equations as,

$$\lambda \ln J + 2\mu \ln \lambda_2 = 0 \tag{5.116a}$$

$$\lambda \ln J + 2\mu \ln \lambda_3 = 0 \tag{5.116b}$$

from which it easily follows that the stretches in the second and third directions are equal and related to the main stretch via Poisson's ratio $\nu$ as,

$$\ln \lambda_2 = \ln \lambda_3 = -\nu \ln \lambda_1; \qquad \nu = \frac{\lambda}{2\lambda + 2\mu} \tag{5.117}$$

Using Equations (5.89–90) and (5.117) a one-dimensional constitutive equation involving the rod stress $\sigma_{11}$, the logarithmic strain $\ln \lambda_1$, and Young's modulus $E$ emerges as,

$$\sigma_{11} = \frac{E}{J} \ln \lambda_1; \qquad E = \frac{\mu(2\mu + 3\lambda)}{\lambda + \mu} \tag{5.118}$$

where $J$ can be obtained with the help of Equation (5.117) in terms of $\lambda_1$ and $\nu$ as,

$$J = \lambda_1^{(1-2\nu)} \tag{5.119}$$

Note that for the incompressible case $J = 1$, Equation (5.118) coincides with the uniaxial constitutive equation employed in Chapter 1.

Finally, the stored elastic energy given by Equation (5.88) becomes,

$$\Psi(\lambda_1) = \frac{E}{2} (\ln \lambda_1)^2 \tag{5.120}$$

and, choosing a local axis in the direction of the rod, the only effective term in the Eulerian tangent modulus $\mathcal{C}_{1111}$ is given by Equation (5.86) as,

$$\mathcal{C}_{1111} = \frac{1}{J}\frac{\partial^2 \Psi}{\partial \ln \lambda_1 \partial \ln \lambda_1} - 2\sigma_{11} = \frac{E}{J} - 2\sigma_{11} \tag{5.121}$$

Again, for the incompressible case $J = 1$, the term $E - 2\sigma_{11}$ was already apparent in Chapter 1 where the equilibrium equation of a rod was linearized in a direct manner.

## Exercises

1. In a plane stress situation the right Cauchy–Green tensor $\boldsymbol{C}$ is,

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}; \qquad C_{33} = \frac{h^2}{H^2}$$

where $H$ and $h$ are the initial and current thickness respectively. Show that incompressibility implies,

$$C_{33} = III_{\overline{C}}^{-1}; \qquad (\boldsymbol{C}^{-1})_{33} = III_{\overline{C}}; \qquad \overline{\boldsymbol{C}} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Using these equations, show that for an incompressible neo-Hookean material the plane stress condition $S_{33} = 0$ enables the pressure in Equation (5.50) to be explicitly evaluated as,

$$p = \tfrac{1}{3}\mu(I_{\overline{C}} - 2III_{\overline{C}}^{-1})$$

and therefore the in-plane components of the second Piola–Kirchhoff and Cauchy tensors are,

$$\overline{\boldsymbol{S}} = \mu(\bar{\boldsymbol{I}} - III_{\overline{C}}\,\overline{\boldsymbol{C}}^{-1})$$

$$\bar{\boldsymbol{\sigma}} = \mu(\bar{\boldsymbol{b}} - III_{\bar{b}}\bar{\boldsymbol{I}})$$

where the overline indicates the $2 \times 2$ components of a tensor.

2. Show that the Equations in Exercise 1 can also be derived by imposing the condition $C_{33} = III_{\overline{C}}^{-1}$ in the neo-Hookean elastic function $\Psi$ to give,

$$\Psi(\overline{\boldsymbol{C}}) = \tfrac{1}{2}\mu(I_{\overline{C}} + III_{\overline{C}}^{-1} - 3)$$

from which $\overline{\boldsymbol{S}}$ is obtained by differentiation with respect to the in-plane tensor $\overline{\boldsymbol{C}}$. Finally, prove that the Lagrangian and Eulerian in-plane elasticity tensors



are,

$$\overline{\mathcal{C}} = 2\mu III_{\overline{C}}^{-1}(\overline{\boldsymbol{C}}^{-1} \otimes \overline{\boldsymbol{C}}^{-1} + \overline{\mathcal{I}})$$

$$\bar{\mathcal{c}} = 2\mu III_{\bar{b}}^{-1}(\bar{\boldsymbol{I}} \otimes \bar{\boldsymbol{I}} + \bar{\boldsymbol{\imath}})$$

3. Using the push back–pull-forward relationships between $\dot{\boldsymbol{E}}$ and $\boldsymbol{d}$ and between $\mathcal{C}$ and $\mathcal{c}$ show that,

$$\dot{\boldsymbol{E}} : \mathcal{C} : \dot{\boldsymbol{E}} = J\boldsymbol{d} : \mathcal{c} : \boldsymbol{d}$$

for any arbitrary motion. Using this equation and recalling Example 5.2, show that,

$$J\mathcal{c} = 4\boldsymbol{b}\frac{\partial^2 \Psi}{\partial \boldsymbol{b}\partial \boldsymbol{b}}\boldsymbol{b}$$

Check that using this equation for the compressible neo-Hookean model you retrieve Equation (5.35).

4. Using the simple stretch-based hyperelastic equations discussed in Section 5.6.5, show that the principal stresses for a simple shear test are,

$$\sigma_{11} = -\sigma_{22} = 2\mu \sinh^{-1}\tfrac{\gamma}{2}$$

Find the Cartesian stress components.

5. A general type of incompressible hyperelastic material proposed by Ogden is defined by the following strain energy function:

$$\Psi = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_2^{\alpha_p} - 3)$$

Derive the homogeneous counterpart of this functional. Obtain expressions for the principal components of the deviatoric stresses and elasticity tensor.

# Appendix

# G

# 15 Numerical Finite and Boundary Element Methods

Reviewing the previous chapters would indicate that analytical solutions to elasticity problems are normally accomplished for regions and loadings with relatively simple geometry. For example, many solutions can be developed for two-dimensional problems, while only a limited number exist for three dimensions. Solutions are commonly available for problems with simple shapes such as those having boundaries coinciding with Cartesian, cylindrical, and spherical coordinate surfaces. Unfortunately, however, problems with more general boundary shape and loading are commonly intractable or require very extensive mathematical analysis and numerical evaluation. Because most real-world problems involve structures with complicated shape and loading, a gap exists between what is needed in applications and what can be solved by analytical closed-form methods.

Over the years, this need to determine deformation and stresses in complex problems has lead to the development of many approximate and numerical solution methods (see brief discussion in Section 5.7). Approximate methods based on energy techniques were outlined in Section 6.7, but it was pointed out that these schemes have limited success in developing solutions for problems of complex shape. Methods of numerical stress analysis normally recast the mathematical elasticity boundary value problem into a direct numerical routine. One such early scheme is the *finite difference method* (FDM) in which derivatives of the governing field equations are replaced by algebraic difference equations. This method generates a system of algebraic equations at various computational grid points in the body, and solution to the system determines the unknown variable at each grid point. Although simple in concept, FDM has not been able to provide a useful and accurate scheme to handle general problems with geometric and loading complexity. Over the past few decades, two methods have emerged that provide necessary accuracy, general applicability, and ease of use. This has lead to their acceptance by the stress analysis community and has resulted in the development of many private and commercial computer codes implementing each numerical scheme.

The first of these techniques is known as the *finite element method* (FEM) and involves dividing the body under study into a number of pieces or subdomains called *elements*. The solution is then approximated over each element and is quantified in terms of values at special locations within the element called the *nodes*. The discretization process establishes an algebraic system of equations for the unknown nodal values, which approximate the continuous solution. Because element size, shape, and approximating scheme can be varied to suit the problem, the method can accurately simulate solutions to problems of complex geometry and loading. FEM has thus become a primary tool for practical stress analysis and is also used extensively in many other fields of engineering and science.

The second numerical scheme, called the *boundary element method* (BEM), is based on an integral statement of elasticity (see relation (6.4.7)). This statement may be cast into a form with unknowns only over the boundary of the domain under study. The boundary integral equation is then solved using finite element concepts where the boundary is divided into elements and the solution is approximated over each element using appropriate interpolation functions. This method again produces an algebraic system of equations to solve for unknown nodal values that approximate the solution. Similar to FEM techniques, BEM also allows variation in element size, shape, and approximating scheme to suit the application, and thus the method can accurately solve a large variety of problems.

Generally, an entire course is required to present sufficient finite and boundary element theory to prepare properly for their numerical/computational application. Thus, the brief presentation in this chapter provides only an overview of each method, focusing on narrow applications for two-dimensional elasticity problems. The primary goal is to establish a basic level of understanding that will allow a quick look at applications and enable connections to be made between numerical solutions (simulations) and those developed analytically in the previous chapters. This brief introduction provides the groundwork for future and more detailed study in these important areas of computational solid mechanics.

## 15.1 Basics of the Finite Element Method

Finite element procedures evolved out of matrix methods used by the structural mechanics community during the 1950s and 1960s. Over the years, extensive research has clearly established and tested numerous FEM formulations, and the method has spread to applications in many fields of engineering and science. FEM techniques have been created for discrete and continuous problems including static and dynamic behavior with both linear and nonlinear response. The method can be applied to one-, two-, or three-dimensional problems using a large variety of standard element types. We, however, limit our discussion to only two-dimensional, linear isotropic elastostatic problems. Numerous texts have been generated that are devoted exclusively to this subject; for example, Reddy (1993), Bathe (1995), Zienkiewicz and Taylor (1989), Fung and Tong (2001), and Cook, Malkus, and Plesha (1989).

As mentioned, the method discretizes the domain under study by dividing the region into subdomains called elements. In order to simplify formulation and application procedures, elements are normally chosen to be simple geometric shapes, and for two-dimensional problems these would be polygons including triangles and quadrilaterals. A two-dimensional example of a rectangular plate with a circular hole divided into triangular elements is shown in Figure 15-1. Two different meshes (discretizations) of the same problem are illustrated, and even at this early stage in our discussion, it is apparent that improvement of the representation is found using the *finer mesh* with a larger number of smaller elements. Within each element, an approximate solution is developed, and this is quantified at particular locations called the nodes. Using a linear approximation, these nodes are located at the vertices of the triangular element as shown in the figure. Other higher-order approximations (quadratic, cubic, etc.) can also be used, resulting in additional nodes located in other positions. We present only a finite element formulation using linear, two-dimensional triangular elements.

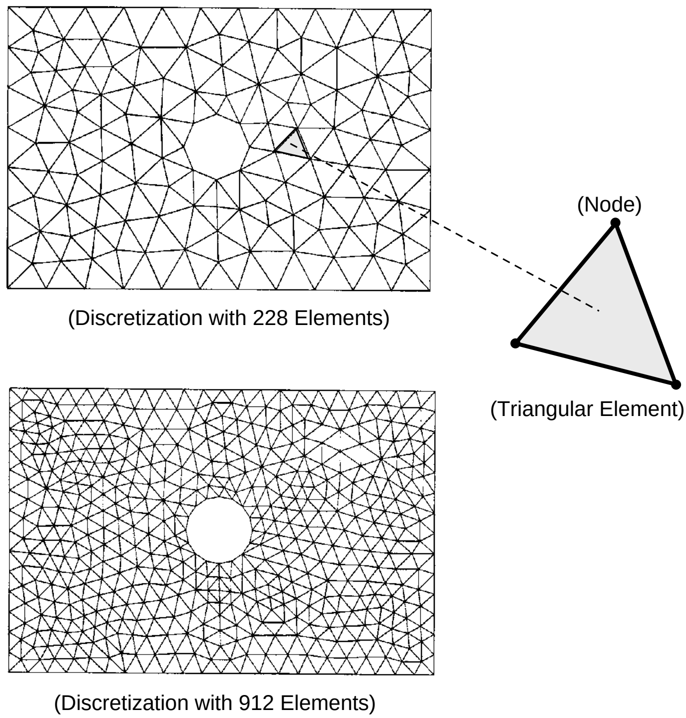



**FIGURE 15-1** *Finite element discretization using triangular elements.*

Typical basic steps in a linear, static finite element analysis include the following:

1. Discretize the body into a finite number of element subdomains
2. Develop approximate solution over each element in terms of nodal values
3. Based on system connectivity, assemble elements and apply all continuity and boundary conditions to develop an algebraic system of equations among nodal values
4. Solve assembled system for nodal values; post process solution to determine additional variables of interest if necessary

The basic formulation of the method lies in developing the element equation that approximately represents the elastic behavior of the element. This development is done for the generic case, thus creating a model applicable to all elements in the mesh. As pointed out in Chapter 6, energy methods offer schemes to develop approximate solutions to elasticity problems, and although these schemes were not practical for domains of complex shape, they can be easily applied over an element domain of simple geometry (i.e., triangle). Therefore, methods of virtual work leading to a Ritz approximation prove to be very useful in developing element equations for FEM elasticity applications. Another related scheme to develop the desired element equation uses a more mathematical approach known as the *method of weighted residuals*. This second technique starts with the governing differential equations, and through appropriate mathematical manipulations, a so-called *weak form* of the system is developed. Using a Ritz/Galerkin scheme, an approximate solution to the weak form is constructed, and this result is identical to the method based on energy and virtual work. Before developing the element equations, we first discuss the necessary procedures to create approximate solutions over an element in the system.

## 15.2 Approximating Functions for Two-Dimensional Linear Triangular Elements

Limiting our discussion to the two-dimensional case with triangular elements, we wish to investigate procedures necessary to develop a linear approximation of a scalar variable $u(x,y)$ over an element. Figure 15-2 illustrates a typical triangular element denoted by $\Omega_e$ in the $x,y$-plane. Looking for a linear approximation, the variable is represented as

$$u(x, y) = c_1 + c_2 x + c_3 y \tag{15.2.1}$$

where $c_i$ are constants. It should be kept in mind that in general the solution variable is expected to have nonlinear behavior over the entire domain and our linear (planar) approximation is only proposed over the element. We therefore are using a piecewise linear approximation to represent the general nonlinear solution over the entire body. This approach generally gives sufficient accuracy if a large number of elements are used to represent the solution field. Other higher-order approximations including quadratic, cubic, and specialized nonlinear forms can also be used to improve the accuracy of the representation.

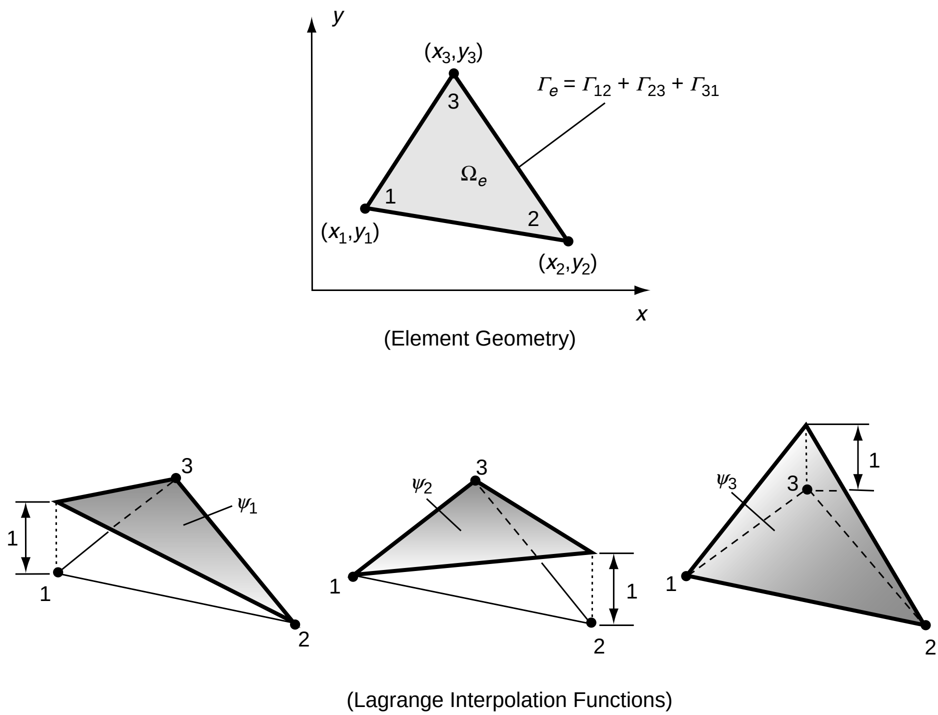



**FIGURE 15-2** *Linear triangular element geometry and interpolation.*

It is normally desired to express the representation (15.2.1) in terms of the nodal values of the solution variable. This can be accomplished by first evaluating the variable at each of the three nodes

$$
\begin{aligned}
u(x_1, y_1) &= u_1 = c_1 + c_2 x_1 + c_3 y_1 \\
u(x_2, y_2) &= u_2 = c_1 + c_2 x_2 + c_3 y_2 \\
u(x_3, y_3) &= u_3 = c_1 + c_2 x_3 + c_3 y_3
\end{aligned}
\tag{15.2.2}
$$

Solving this system of algebraic equations, the constants $c_i$ can be expressed in terms of the nodal values $u_i$, and the general results are given by

$$
\begin{aligned}
c_1 &= \frac{1}{2A_e}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \\
c_2 &= \frac{1}{2A_e}(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3) \\
c_3 &= \frac{1}{2A_e}(\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)
\end{aligned}
\tag{15.2.3}
$$

where $A_e$ is the area of the element, and $\alpha_i = x_j y_k - x_k y_j$, $\beta_i = y_j - y_k$, $\gamma_i = x_k - x_j$, where $i \neq j \neq k$ and $i,j,k$ permute in natural order. Substituting for $c_i$ in (15.2.1) gives

$$
\begin{aligned}
u(x, y) &= \frac{1}{2A_e}[(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \\
&\quad + (\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3)x \\
&\quad + (\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)y] \\
&= \sum_{i=1}^{3} u_i \psi_i(x, y)
\end{aligned}
\tag{15.2.4}
$$

where $\psi_i$ are the *interpolation functions* for the triangular element given by

$$
\psi_i(x, y) = \frac{1}{2A_e}(\alpha_i + \beta_i x + \gamma_i y) \tag{15.2.5}
$$

It is noted that the form of the interpolation functions depends on the initial approximation assumption and on the shape of the element. Each of the three interpolation functions represents a planar surface as shown Figure 15-2, and it is observed that they will satisfy the following conditions:

$$
\psi_i(x_j, y_j) = \delta_{ij}, \ \sum_{i=1}^{3} \psi_i = 1 \tag{15.2.6}
$$

Functions satisfying such conditions are referred to as *Lagrange interpolation functions*.

This method of using interpolation functions to represent the approximate solution over an element quantifies the approximation in terms of nodal values. In this fashion, the continuous solution over the entire problem domain is represented by discrete values at particular nodal locations. This discrete representation can be used to determine the solution at other points in the region using various other interpolation schemes. With these representation concepts established, we now pursue a brief development of the plane elasticity element equations using the virtual work formulation.

## 15.3 Virtual Work Formulation for Plane Elasticity

The principle of virtual work developed in Section 6.5 can be stated over a finite element volume $V_e$ with boundary $S_e$ as

$$\int_{V_e} \sigma_{ij}\delta e_{ij} dV = \int_{S_e} T_i^n \delta u_i dS + \int_{V_e} F_i \delta u_i dV \tag{15.3.1}$$

For plane elasticity with an element of uniform thickness $h_e$, $V_e = h_e\Omega_e$ and $S_e = h_e\Gamma_e$, and the previous relation can be reduced to the two-dimensional form

$$\begin{aligned} h_e &\int_{\Omega_e} (\sigma_x\delta e_x + \sigma_y\delta e_y + 2\tau_{xy}\delta e_{xy})dxdy \\ &- h_e\int_{\Gamma_e} (T_x^n\delta u + T_y^n\delta v)ds - h_e\int_{\Omega_e} (F_x\delta u + F_y\delta v)dxdy = 0 \end{aligned} \tag{15.3.2}$$

Using matrix notation, this relation can be written as

$$\begin{aligned} h_e &\int_{\Omega_e} \left( \begin{Bmatrix} \delta e_x \\ \delta e_y \\ 2\delta e_{xy} \end{Bmatrix}^T \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \right) dxdy \\ - h_e &\int_{\Gamma_e} \left( \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} \right) ds - h_e\int_{\Omega_e} \left( \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \right) dxdy = 0 \end{aligned} \tag{15.3.3}$$

We now proceed to develop an element formulation in terms of the displacements and choose a linear approximation for each component

$$\begin{aligned} u(x,y) &= \sum_{i=1}^{3} u_i\psi_i(x,y) \\ v(x,y) &= \sum_{i=1}^{3} v_i\psi_i(x,y) \end{aligned} \tag{15.3.4}$$

where $\psi_i(x,y)$ are the Lagrange interpolation functions given by (15.2.5). Using this scheme there will be two unknowns or *degrees of freedom* at each node, resulting in a total of six degrees of freedom for the linear triangular element. Because the strains are related to displacement gradients, this interpolation choice results in a constant strain element (CST), and of course the stresses will also be element-wise constant. Relation (15.3.4) can be expressed in matrix form:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \psi_1 & 0 & \psi_2 & 0 & \psi_3 & 0 \\ 0 & \psi_1 & 0 & \psi_2 & 0 & \psi_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = [\boldsymbol{\psi}]\{\boldsymbol{\Delta}\} \tag{15.3.5}$$

The strains can then be written as

$$\begin{aligned} \{\boldsymbol{e}\} = \begin{Bmatrix} e_x \\ e_y \\ 2e_{xy} \end{Bmatrix} &= \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \\ &= \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} [\boldsymbol{\psi}]\{\boldsymbol{\Delta}\} = [\boldsymbol{B}]\{\boldsymbol{\Delta}\} \end{aligned} \tag{15.3.6}$$

where

$$[\boldsymbol{B}] = \begin{bmatrix} \dfrac{\partial\psi_1}{\partial x} & 0 & \dfrac{\partial\psi_2}{\partial x} & 0 & \dfrac{\partial\psi_3}{\partial x} & 0 \\ 0 & \dfrac{\partial\psi_1}{\partial y} & 0 & \dfrac{\partial\psi_2}{\partial y} & 0 & \dfrac{\partial\psi_3}{\partial y} \\ \dfrac{\partial\psi_1}{\partial y} & \dfrac{\partial\psi_1}{\partial x} & \dfrac{\partial\psi_2}{\partial y} & \dfrac{\partial\psi_2}{\partial x} & \dfrac{\partial\psi_3}{\partial y} & \dfrac{\partial\psi_3}{\partial x} \end{bmatrix} = \frac{1}{2A_e} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \tag{15.3.7}$$

Hooke's law then takes the form

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{C}]\{\boldsymbol{e}\} = [\boldsymbol{C}][\boldsymbol{B}]\{\boldsymbol{\Delta}\} \tag{15.3.8}$$

where $[\boldsymbol{C}]$ is the elasticity matrix that can be generalized to the orthotropic case (see Section 11.2) by

$$[\boldsymbol{C}] = \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} & 0 \\ \boldsymbol{C}_{12} & \boldsymbol{C}_{22} & 0 \\ 0 & 0 & \boldsymbol{C}_{66} \end{bmatrix} \tag{15.3.9}$$

For isotropic materials,

$$\begin{aligned} C_{11} = C_{22} &= \begin{cases} \dfrac{E}{1-v^2} \cdots \text{ plane stress} \\ \dfrac{\mathrm{E}(1-v)}{(1+v)(1-2v)} \cdots \text{plane strain} \end{cases} \\ C_{12} &= \begin{cases} \dfrac{Ev}{1-v^2} \cdots \text{plane stress} \\ \dfrac{\mathrm{E}v}{(1+v)(1-2v)} \cdots \text{ plane strain} \end{cases} \\ C_{66} = \mu &= \frac{E}{2(1+v)} \cdots \text{plane stress and plane strain} \end{aligned} \tag{15.3.10}$$

Using results (15.3.5), (15.3.6), and (15.3.8) in the virtual work statement (15.3.3) gives

$$
\begin{aligned}
&h_e \int_{\Omega_e} \{\delta \boldsymbol{\Delta}\}^T([\boldsymbol{B}]^T[\boldsymbol{C}][\boldsymbol{B}])\{\boldsymbol{\Delta}\} dxdy \\
&- h_e \int_{\Omega_e} \{\delta \boldsymbol{\Delta}\}^T[\boldsymbol{\psi}]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} dxdy - h_e \int_{\Gamma_e} \{\delta \boldsymbol{\Delta}\}^T[\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds = 0
\end{aligned}
\tag{15.3.11}
$$

which can be written in compact form

$$
\{\delta \boldsymbol{\Delta}\}^T([\boldsymbol{K}]\{\boldsymbol{\Delta}\} - \{\boldsymbol{F}\} - \{\boldsymbol{Q}\}) = 0 \tag{15.3.12}
$$

Because this relation is to hold for arbitrary variations $\{\delta \boldsymbol{\Delta}\}^T$, the expression in parentheses must vanish, giving the finite element equation

$$
[\boldsymbol{K}]\{\boldsymbol{\Delta}\} = \{\boldsymbol{F}\} + \{\boldsymbol{Q}\} \tag{15.3.13}
$$

The equation matrices are defined as follows:

$$
\begin{aligned}
[\boldsymbol{K}] &= h_e \int_{\Omega_e} [\boldsymbol{B}]^T[\boldsymbol{C}][B] dxdy \cdots \text{Stiffness Matrix} \\
\{\boldsymbol{F}\} &= h_e \int_{\Omega_e} [\boldsymbol{\psi}]^{\mathbf{T}} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} dxdy \cdots \text{Body Force Vector} \\
\{\boldsymbol{Q}\} &= h_e \int_{\Gamma_e} [\boldsymbol{\psi}]^{\mathbf{T}} \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds \cdots \text{ Loading Vector}
\end{aligned}
\tag{15.3.14}
$$

Using the specific interpolation functions for the constant strain triangular element, the $[B]$ matrix had constant components given by (15.3.7). If we assume that the elasticity matrix also does not vary over the element, then the stiffness matrix is given by

$$
[\boldsymbol{K}] = h_e A_e [\boldsymbol{B}]^T[\boldsymbol{C}][\boldsymbol{B}] \tag{15.3.15}
$$

and multiplying out the matrices gives the specific form

$$
[\boldsymbol{K}] = \frac{h_e}{4A_e} \begin{bmatrix}
\beta_1^2 C_{11} + \gamma_1^2 C_{66} & \beta_1\gamma_1 C_{12} + \beta_1\gamma_1 C_{66} & \beta_1\beta_2 C_{11} + \gamma_1\gamma_2 C_{66} & \beta_1\gamma_2 C_{12} + \beta_2\gamma_1 C_{66} & \beta_1\beta_3 C_{11} + \gamma_1\gamma_3 C_{66} & \beta_1\gamma_3 C_{12} + \beta_3\gamma_1 C_{66} \\
\cdot & \gamma_1^2 C_{22} + \beta_1^2 C_{66} & \beta_2\gamma_1 C_{12} + \beta_1\gamma_2 C_{66} & \gamma_1\gamma_2 C_{22} + \beta_1\beta_2 C_{66} & \beta_3\gamma_1 C_{12} + \beta_1\gamma_3 C_{66} & \gamma_1\gamma_3 C_{22} + \beta_1\beta_3 C_{66} \\
\cdot & \cdot & \beta_2^2 C_{11} + \gamma_2^2 C_{66} & \beta_2\gamma_2 C_{12} + \beta_2\gamma_2 C_{66} & \beta_2\beta_3 C_{11} + \gamma_2\gamma_3 C_{66} & \beta_2\gamma_3 C_{12} + \beta_3\gamma_2 C_{66} \\
\cdot & \cdot & \cdot & \gamma_2^2 C_{22} + \beta_2^2 C_{66} & \beta_3\gamma_2 C_{12} + \beta_2\gamma_3 C_{66} & \gamma_2\gamma_3 C_{22} + \beta_2\beta_3 C_{66} \\
\cdot & \cdot & \cdot & \cdot & \beta_3^2 C_{11} + \gamma_3^2 C_{66} & \beta_3\gamma_3 C_{12} + \beta_3\gamma_3 C_{66} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \gamma_3^2 C_{22} + \beta_3^2 C_{66}
\end{bmatrix} \tag{15.3.16}
$$

Note that the stiffness matrix is always symmetric, and thus only the top-right (or bottom-left) portion need be explicitly written out. If we also choose body forces that are element-wise constant, the body force vector $\{\boldsymbol{F}\}$ can be integrated to give

$$
\{\boldsymbol{F}\} = \frac{h_e A_e}{3} \{F_x \ F_y \ F_x \ F_y \ F_x \ F_y\}^T \tag{15.3.17}
$$

The $\{\boldsymbol{Q}\}$ matrix involves integration of the tractions around the element boundary, and its evaluation depends on whether an element side falls on the boundary of the domain or is located in the region's interior. The evaluation also requires a modeling decision on the assumed traction variation on the element sides. Most problems can be adequately modeled using constant, linear, or quadratic variation in the element boundary tractions. For the typical triangular element shown in Figure 15-2, the $\{\boldsymbol{Q}\}$ matrix may be written as

$$\begin{aligned}\{\boldsymbol{Q}\} &= h_e \int_{\Gamma} [\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds \\ &= h_e \int_{\Gamma_{12}} [\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds + h_e \int_{\Gamma_{23}} [\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds + h_e \int_{\Gamma_{31}} [\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds\end{aligned} \tag{15.3.18}$$

Wishing to keep our study brief in theory, we take the simplest case of element-wise constant boundary tractions, which allows explicit calculation of the boundary integrals. For this case, the integral over element side $\Gamma_{12}$ is given by

$$h_e \int_{\Gamma_{12}} [\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds = h_e \int_{\Gamma_{12}} \begin{Bmatrix} \psi_1 T_x^n \\ \psi_1 T_y^n \\ \psi_2 T_x^n \\ \psi_2 T_y^n \\ \psi_3 T_x^n \\ \psi_3 T_y^n \end{Bmatrix} ds = \frac{h_e L_{12}}{2} \begin{Bmatrix} T_x^n \\ T_y^n \\ T_x^n \\ T_y^n \\ 0 \\ 0 \end{Bmatrix}_{12} \tag{15.3.19}$$

where $L_{12}$ is the length of side $\Gamma_{12}$. Note that we have used the fact that along side $\Gamma_{12}$, $\psi_1$ and $\psi_2$ vary linearly and $\psi_3 = 0$. Following similar analysis, the boundary integrals along sides $\Gamma_{23}$ and $\Gamma_{31}$ are found to be

$$h_e \int_{\Gamma_{23}} [\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds = \frac{h_e L_{23}}{2} \begin{Bmatrix} 0 \\ 0 \\ T_x^n \\ T_y^n \\ T_x^n \\ T_y^n \end{Bmatrix}_{23}, \quad h_e \int_{\Gamma_{31}} [\boldsymbol{\psi}]^T \begin{Bmatrix} T_x^n \\ T_y^n \end{Bmatrix} ds = \frac{h_e L_{31}}{2} \begin{Bmatrix} T_x^n \\ T_y^n \\ 0 \\ 0 \\ T_x^n \\ T_y^n \end{Bmatrix}_{31} \tag{15.3.20}$$

It should be noted that for element sides that lie in the region's interior, values of the boundary tractions will not be known before the solution is found, and thus the previous relations cannot be used to evaluate the contributions of the $\{\boldsymbol{Q}\}$ matrix explicitly. However, for this situation, the stresses and tractions are in internal equilibrium, and thus the integrated result from one element will cancel that from the opposite adjacent element when the finite element system is assembled. For element sides that coincide with the region's boundary, any applied boundary tractions are then incorporated into the results given by relations (15.3.19) and (15.3.20). Our simplifications of choosing element-wise constant values for the elastic moduli, body forces, and tractions were made only for convenience of the current abbreviated presentation. Normally, FEM modeling allows considerably more generality in these choices and integrals in the basic element equation (15.3.14) are then evaluated numerically for such applications.

## 15.4 FEM Problem Application

Applications using the linear triangular element discretize the domain into a connected set of such elements; see, for example, Figure 15-1. The mesh geometry establishes which elements are interconnected and identifies those on the boundary of the domain. Using computer implementation, each element in the mesh is mapped or transformed onto a master element in a local coordinate system where all calculations are done. The overall problem is then modeled by assembling the entire set of elements through a process of invoking equilibrium at each node in the mesh. This procedure creates a global assembled matrix system equation of similar form as (15.3.13). Boundary conditions are then incorporated into this global system to reduce the problem to a solvable set of algebraic equations for the unknown nodal displacements. We do not pursue the theoretical and operational details in these procedures, but rather focus attention on a particular example to illustrate some of the key steps in the process.

### EXAMPLE 15-1: Elastic Plate Under Uniform Tension

Consider the plane stress problem of an isotropic elastic plate under uniform tension with zero body forces as shown in Figure 15-3. For convenience, the plate is taken with unit dimensions and thickness and is discretized into two triangular elements as shown. This simple problem is chosen in order to demonstrate some of the basic FEM solution procedures previously presented. More complex examples are discussed in the next section to illustrate the general power and utility of the numerical technique.

The element mesh is labeled as shown with local node numbers within each element and global node numbers (1–4) for the entire problem. We start by developing the equation for each element and then assemble the two elements to model the entire plate. For element 1, the geometric parameters are $\beta_1 = -1,\ \beta_2 = 1,\ \beta_3 = 0,\ \gamma_1 = 0,\ \gamma_2 = -1,\ \gamma_3 = 1$, and $A_1 = 1/2$. For the isotropic plane stress case, the element equation follows from our previous work:

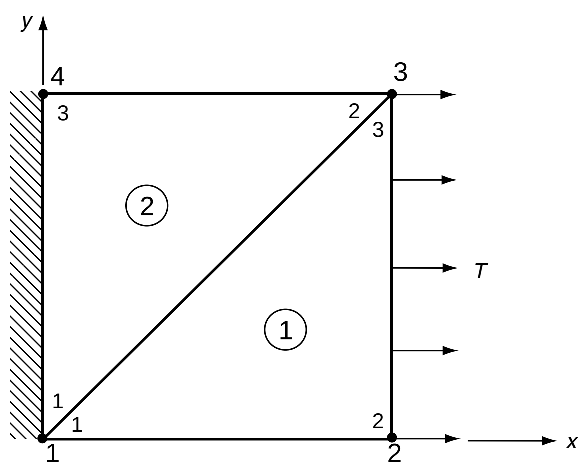



**FIGURE 15-3** *FEM analysis of elastic plate under uniform tension.*

$$\frac{E}{2(1-\nu^2)}\begin{bmatrix} 1 & 0 & -1 & \nu & 0 & -\nu \\ \cdot & \frac{1-\nu}{2} & \frac{1-\nu}{2} & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & 0 \\ \cdot & \cdot & \frac{3-\nu}{2} & -\frac{1+\nu}{2} & -\frac{1-\nu}{2} & \nu \\ \cdot & \cdot & \cdot & \frac{3-\nu}{2} & \frac{1-\nu}{2} & -1 \\ \cdot & \cdot & \cdot & \cdot & \frac{1-\nu}{2} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}\begin{Bmatrix} u_1^{(1)} \\ v_1^{(1)} \\ u_2^{(1)} \\ v_2^{(1)} \\ u_3^{(1)} \\ v_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} T_{1x}^{(1)} \\ T_{1y}^{(1)} \\ T_{2x}^{(1)} \\ T_{2y}^{(1)} \\ T_{3x}^{(1)} \\ T_{3y}^{(1)} \end{Bmatrix} \tag{15.4.1}$$

In similar fashion for element 2, $\beta_1 = 0,\ \beta_2 = 1,\ \beta_3 = -1,\ \gamma_1 = -1,\ \gamma_2 = 0,\ \gamma_3 = 1,\ A_1 = 1/2$, and the element equation becomes

$$\frac{E}{2(1-\nu^2)}\begin{bmatrix} \frac{1-\nu}{2} & 0 & 0 & -\frac{1-\nu}{2} & -\frac{1-\nu}{2} & \frac{1-\nu}{2} \\ \cdot & 1 & -\nu & 0 & \nu & -1 \\ \cdot & \cdot & 1 & 0 & -1 & \nu \\ \cdot & \cdot & \cdot & \frac{1-\nu}{2} & \frac{1-\nu}{2} & -\frac{1-\nu}{2} \\ \cdot & \cdot & \cdot & \cdot & \frac{3-\nu}{2} & -\frac{1-\nu}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{3-\nu}{2} \end{bmatrix}\begin{Bmatrix} u_1^{(2)} \\ v_1^{(2)} \\ u_2^{(2)} \\ v_2^{(2)} \\ u_3^{(2)} \\ v_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} T_{1x}^{(2)} \\ T_{1y}^{(2)} \\ T_{2x}^{(2)} \\ T_{2y}^{(2)} \\ T_{3x}^{(2)} \\ T_{3y}^{(2)} \end{Bmatrix} \tag{15.4.2}$$

These individual element equations are to be assembled to model the plate, and this is carried out using the global node numbering format by enforcing $x$ and $y$ equilibrium at each node. The final result is given by the assembled global system

$$\begin{bmatrix} K_{11}^{(1)}+K_{11}^{(2)} & K_{12}^{(1)}+K_{12}^{(2)} & K_{13}^{(1)} & K_{14}^{(1)} & K_{15}^{(1)}+K_{13}^{(2)} & K_{16}^{(1)}+K_{14}^{(2)} & K_{15}^{(1)} & K_{16}^{(1)} \\ \cdot & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(1)} & K_{24}^{(1)} & K_{25}^{(1)}+K_{23}^{(2)} & K_{26}^{(1)}+K_{24}^{(2)} & K_{25}^{(1)} & K_{26}^{(1)} \\ \cdot & \cdot & K_{33}^{(1)} & K_{34}^{(1)} & K_{35}^{(1)} & K_{36}^{(1)} & 0 & 0 \\ \cdot & \cdot & \cdot & K_{44}^{(1)} & K_{45}^{(1)} & K_{46}^{(1)} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & K_{55}^{(1)}+K_{33}^{(2)} & K_{56}^{(1)}+K_{34}^{(2)} & K_{35}^{(2)} & K_{36}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & K_{66}^{(1)}+K_{44}^{(2)} & K_{45}^{(2)} & K_{46}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & K_{55}^{(2)} & K_{56}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & K_{66}^{(2)} \end{bmatrix} \tag{15.4.3}$$

$$\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} T_{1x}^{(1)}+T_{1x}^{(2)} \\ T_{1y}^{(1)}+T_{1y}^{(2)} \\ T_{2x}^{(1)} \\ T_{2y}^{(1)} \\ T_{3x}^{(1)}+T_{2x}^{(2)} \\ T_{3y}^{(1)}+T_{2y}^{(2)} \\ T_{3x}^{(2)} \\ T_{3y}^{(2)} \end{Bmatrix}$$

*Continued*

## EXAMPLE 15-1: Elastic Plate Under Uniform Tension–*Cont'd*

where $U_i$ and $V_i$ are the global $x$ and $y$ nodal displacements, and $K_{ij}^{(1)}$ and $K_{ij}^{(2)}$ are the local stiffness components for elements 1 and 2 as given in relations (15.4.1) and (15.4.2).

The next step is to use the problem boundary conditions to reduce this global system. Because the plate is fixed along its left edge, $U_1 = V_1 = U_4 = V_4 = 0$. Using the scheme presented in equations (15.3.18) through (15.3.20), the tractions on the right edge are modeled by choosing $T_{2x}^{(1)} = T/2$, $T_{2y}^{(1)} = 0$, $T_{3x}^{(1)} + T_{2x}^{(2)} = T/2$, $T_{3y}^{(1)} + T_{2y}^{(2)} = 0$. These conditions reduce the global system to

$$\begin{bmatrix} K_{33}^{(1)} & K_{34}^{(1)} & K_{35}^{(1)} & K_{36}^{(1)} \\ \cdot & K_{44}^{(1)} & K_{45}^{(1)} & K_{46}^{(1)} \\ \cdot & \cdot & K_{55}^{(1)} + K_{33}^{(2)} & K_{56}^{(1)} + K_{34}^{(2)} \\ \cdot & \cdot & \cdot & K_{66}^{(1)} + K_{44}^{(2)} \end{bmatrix} \begin{Bmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} T/2 \\ 0 \\ T/2 \\ 0 \end{Bmatrix} \tag{15.4.4}$$

This result can be then be solved for the nodal unknowns, and for the case of material with properties $E = 207\,\text{GPa}$ and $v = 0.25$, the solution is found to be

$$\begin{Bmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0.492 \\ 0.081 \\ 0.441 \\ -0.030 \end{Bmatrix} T \times 10^{-11} m \tag{15.4.4}$$

Note that the FEM displacements are not symmetric as expected from analytical theory. This is caused by the fact that our simple two-element discretization eliminated the symmetry in the original problem. If another symmetric mesh were used, the displacements at nodes 2 and 3 would then be symmetric. As a postprocessing step, the forces at nodes 1 and 4 could now be computed by back-substituting solution (15.4.4) into the general equation (15.4.3). Many of the basic steps in an FEM solution are demonstrated in this hand-calculation example. However, the importance of the numerical method lies in its computer implementation, and examples of this are now discussed.

# 15.5 FEM Code Applications

The power and utility of the finite element method lies in the use of computer codes that implement the numerical method for problems of general shape and loading. A very large number of both private and commercial FEM computer codes have been developed over the past few decades. Many of these codes (e.g., ABAQUS, ANSYS, ALGOR, NASTRAN, ADINA) offer very extensive element libraries and can handle linear and nonlinear problems under either static or dynamic conditions. However, the use of such general codes requires considerable study and practice and would not suit the needs of this chapter. Therefore, rather than attempting to use a general code, we follow our numerical theme of employing MATLAB software, which offers a simple FEM package appropriate for our limited needs. The MATLAB code is called the *PDE Toolbox* and is one of the many toolboxes distributed with the basic software. This software package provides an FEM code that can solve two-dimensional elasticity problems using linear triangular elements. Additional problems governed by other partial differential equations can also be handled, and this allows the software to also be used for the torsion problem. The PDE Toolbox